Numerical simulations of shoaling internal solitary waves in tilting tank experiments

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1. Introduction

Internal solitary wave (ISW) trains have been observed in many lakes (e.g., Farmer 1978. See also references in Boegman et al. 2005 and Vlasenko & Hutter 2002). They are indirectly generated by winds which blow warm surface water to the lee of the lake creating a tilted thermocline. When the wind dies down basin scale internal waves start propagating. Nonlinearity and dispersion results in the formation of the ISW trains. An important problem in physical limnology is understanding the energy content of these high-frequency waves and the consequences of their breaking when they reach the shore. These waves are inherently non-hydrostatic and of short wave length making them difficult to simulate in lake-scale numerical models. The high resolutions and large domains required to simulate the formation, run-up and breaking of solitary wave makes the use of a two-dimensional model highly desirable. Two-dimensional models are currently the norm in such simulations (Fringer & Street 2003, Vlasenko & Hutter 2002). This presents problems if one of the objectives is to predict the amount of mixing associated with breaking ISWs, an inherently three-dimensional process.

In order to elucidate the breaking of shoaling ISWs Boegman \textit{et al.} (2005) did a series of laboratory experiments using a tilting tank. In this paper some preliminary results of two-dimensional non-hydrostatic simulations of one of these experiments are presented. Comparisons of the amount of mixing predicted by the simulations with that in the laboratory experiments is made.

2. Experiments

The experiments were carried out in a sealed acrylic tank 6 m long, 0.3 m wide and 0.29 m deep. The tank was filled with a two-layer stratification using fresh and saline water. A sloping bottom with a slope of 0.1 or 0.15 was placed at one end of the tank. After filling with water the tank was rotated by a small angle and the fluid was allowed to come to rest. Then, the tank was rapidly rotated to the horizontal. This resulted in an initial state with a sloping pycnocline. The adjustment of this
state was measured with three ultrasonic wave guages (Michallet & Berthélemly 1997) and using a digital camera (Boegman et al. 2005).

3. The numerical model

The numerical model used in this study is a modified version of the non-hydrostatic model described in Lamb (1994), which uses the Boussinesq approximation. The model uses a rigid lid with the surface at $z = 0$. The rotation of the Earth is ignored due to the short length and time scales considered. The effects of rotation of the tank about the horizontal $y$ axis are included because the fluid in the tank rotates relative to the tank as the tank is brought to the horizontal at the
beginning of the experiment. The model equations are

\[
\frac{D\vec{U}}{Dt} + 2 \frac{d\theta}{dt} (w, -u) = -\vec{\nabla} p - \rho g (\sin \theta, \cos \theta) - \frac{d^2 \theta}{dt^2} (z, -x) + \vec{F}_u \tag{1}
\]

\[
\frac{D\rho}{Dt} = F_\rho, \tag{2}
\]

\[
\vec{\nabla} \cdot \vec{U} = 0, \tag{3}
\]

where \(\vec{U} = (u, w)\) is the velocity vector in a reference frame fixed with the tank, \(u\) and \(w\) being the horizontal (\(x\)) and vertical (\(z\)) components.

The scaled density and pressure \(\rho\) and \(p\) are related to the fluid density and pressure \(\rho^*\) and \(p^*\) via \(\rho^* = \rho_o (1 + \rho)\) and \(p^* = \rho_o (-gz + p)\) where \(\rho_o\) is the reference density. The angle made by the along-tank \(x\) axis from the horizontal is \(\theta(t)\) is measured counterclockwise about the negative \(y\)-axis. The centripetal acceleration \(\vec{\Omega} \times (\vec{\Omega} \times \vec{x})\), where \(\vec{\Omega} = (0, -d\theta/dt, 0)\), which has the form of a gradient, has been absorbed into the pressure. The viscous and diffusive terms have the form

\[
\vec{F}_u = \left(K_u \vec{U}_z\right)_z + K_{hu} \vec{U}_{z\chi} \tag{4}
\]

\[
\vec{F}_p = \left(K_\rho \rho_z\right)_z + K_{h\rho} \rho_{z\chi} \tag{5}
\]

Terrain following (sigma) coordinates are used and \(\chi\) is the horizontal terrain-following coordinate. No-slip boundary conditions are used for the velocity field along the bottom and top boundaries (the tank is completely filled with fluid) and along the end walls. No-flux boundary conditions are used for the density.

Because terrain following coordinates are used the fluid depth cannot be reduced to zero at the left end of the tank as in the laboratory situation. The bottom of the tank is at

\[
z = -0.02 - 0.5m \left( \tanh(x, x_1, d) - \tanh(x, x_2, d) \right), \tag{6}
\]

where

\[
\tanh(x_k, a, s) = \int_{-\infty}^{x_k} 1 + \tanh \left( \frac{x_k' - a}{s} \right) dx_k',
\]

is a function which smoothly changes from 0 to a constant slope of 2 at \(x = a\) over a characteristic distance \(s\). For these simulations

\[
x_1 = 0.02/m, \tag{7}
\]

\[
x_2 = x_1 + 0.27/m, \tag{8}
\]

\[
d = 0.05, \tag{9}
\]
Figure 2: Onset of first breaking event at $t = 84$. Upper panel shows the density field, lower panel is the horizontal velocity field with solid/dashed contours indicating positive/negative velocities. Case 11.

where the slope of the sloping part of the bottom is

$$m = \frac{0.29}{l},$$

(10)

where $l = 2$ or $3$ m is the length of the sloping bottom in the laboratory experiment. All terms in these expressions have units of meters.

The initial stratification has the form

$$\rho = \frac{\delta \rho}{2} \tanh \frac{z - \gamma x - z_{pyc}}{d_{pyc}}.$$

(11)

Here $\delta \rho$ is the density jump across the pycnocline, scaled by the reference density $\rho_o$, $d_{pyc}$ gives the thickness of the pycnocline and $\gamma$ is the initial slope of the pycnocline relative to the along-tank axis.

The laboratory experiments begin with the tank, initially inclined at an angle $\theta_o$ to the vertical, being rapidly brought to the horizontal. Because the tank is brought to the horizontal in a few seconds baroclinic generation of vorticity is negligible and hence the fluid, which is initially irrotational, remains irrotational outside of the viscous boundary layers adjacent to the tank boundaries. As a consequence, the fluid in the tank rotates relative to the tank as the tank is brought to the horizontal as if the density is constant. The movement of the fluid in the tank can be estimated on this basis. For an angle $\theta_o = 1.56^\circ = 0.0274$ rad corresponding to experiment 14.09.04 which is discussed below, the fluid moves about $0.75$ cm along the upper and lower boundaries except within about $30$ cm (tank depth) of the end walls, and it moves about $0.5$ cm along the end walls at the mid-depth.
Table 1: Subsample of some of the simulations of experiment 14.09.04. \( I \) and \( J \) are the number of grid points in the horizontal and vertical, \( \Delta t \) is the maximum time step allowed, \( K_u \) and \( K_{\rho} \) are the vertical viscosity and diffusivity, \( K_{h_{\rho}} \) is the horizontal diffusivity. The horizontal viscosity was equal to the vertical viscosity in all of these runs.

<table>
<thead>
<tr>
<th>case</th>
<th>( I )</th>
<th>( J )</th>
<th>( \max \Delta t ) (s)</th>
<th>( K_u ) ( \times 10^{-6} \text{ m s}^{-1} )</th>
<th>( K_{\rho} ) ( \times 10^{-6} \text{ m s}^{-1} )</th>
<th>( K_{h_{\rho}} ) ( \times 10^{-6} \text{ m s}^{-1} )</th>
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<td>0.0078125</td>
<td>2.0</td>
<td>0.01</td>
<td>0.01</td>
</tr>
</tbody>
</table>

3. Results: Experiment Exp14.09.04

This experiment begins with downwelling over the slope. The initial density field has the form (13) with \( \delta \rho = 0.0203 \), \( \gamma = 0.0274 \), \( z_{pyc} = -0.0871 \text{ m} \) and \( d_{pyc} = 0.0079 \text{ m} \). The tank lies between \( x = -3 \text{ m} \) and \( x = 3 \text{ m} \) with the surface at \( z = 0 \). For the runs discussed here the tank was not rotated, i.e., \( \theta = 0 \) in equation (1), as the effects of rotating the tank were found to be negligible.

Figure 1 shows a sequence of density contour plots from 0 to 60 s illustrating the formation of a solitary wave train. The vertical dashed lines in the figures indicate the location of two of the wave gauges which were used to measure the height of the pycnocline in the laboratory experiments. In the left half of the tank the pycnocline thins as it rises. At the right end of the tank the pycnocline thickens. This is largely due to the flow in the corner: an inviscid simulation gave a similar result. Figure 2 shows the onset of the first breaking event at time \( t = 84 \text{ s} \). The density field is shown in the upper panel while the horizontal velocity field is shown in the lower panel. A small separation bubble along the bottom boundary can be seen immediately below the breaking wave.

A large number of model runs have been done with different resolutions, time steps, viscosities and diffusivities. Details of six of these are presented in table 1. The time evolution of the background potential energy (BPE) for these cases is shown in figure 3. The BPE is obtained by sorting the density field to obtain a horizontally uniform background density \( \rho_r(z) \) and computing the potential energy of the sorted density field. The increasing BPE corresponds to a permanent thickening of the pycnocline. In the figure it can be seen that the BPE increases very slowly for
the first 20–40 s. For comparison a line indicating the theoretical increase in BPE for a rest state, assuming a flat bottom, is shown for a diffusivity of $10^{-7}$ m$^2$ s$^{-1}$. This is approximately 10 times larger than the diffusivity of salt. Thereafter the growth rate increases until $t \approx 90$ s after which it grows very rapidly. This rapid growth corresponds to the first breaking event during which several solitary wave break on the slope. A second, weaker, breaking event occurs at about $t = 200$ s.

The only difference between cases 5 and 10 is in the vertical diffusivity. Lowering the diffusivity from $10^{-7}$ to $10^{-8}$ m$^2$ s$^{-1}$ significantly lowers the increase in BPE in the first breaking event. Comparing cases 10 and 11, which differ only in the resolution, shows that increasing the resolution also decreases the amount mixing in the first breaking event and that case 10 is under resolved. Case 11 took about one week to run making longer or higher resolutions runs difficult. Cases 10, 15 and 16 differ only in the time step. The latter two cases, with the smaller time steps are similar. Cases 17 and 10 differ in the horizontal diffusivity $K h_\rho$, with $K h_\rho$ being 200 times larger in case 10. Comparison of these two cases indicates that the results are not very sensitive to $K h_\rho$.

Figure 4 compares the density profiles measured before and after the laboratory experiment with the sorted density profiles $\rho_r(z)$ from the numerical simulations. The initial experimental and numerical profiles are slightly different. Results from two cases (11 and 17) are shown. Both indicate that the final pycnocline is a bit thicker than in the experimental results, particularly as the experimental measurement was taken after the water in the tank was at rest. Note, however, that for case 17 the density profiles at $t = 300$ and 400 s are very similar. Little further thickening can be expected.
Figure 4: Comparison of density profiles for experiment 14.09.04. Only a portion of the water column is shown. Solid curves are lab measurements taken before and after the experiment. Dotted curves are numerical results every 100 s starting at $t = 0$. (a) Case 11, numerical results at $t = 0$, 100 and 200 s. (b) Case 17, numerical results at $t = 0$, 100, 200, 300 and 400 s.

Figure 5 compares the interface displacement (relative to the undisturbed location of the pycnocline when the tank is level) measured in the laboratory (solid) and in the numerical experiments (dotted) at two different locations. Panels (a) and (b) show the results at $x = 0.07$ m for cases 11 and 17 respectively. Panels (c) and (d) are the results at $x = 1.67$ m. Case 17, which ran for 400 s, shows that the general decay of the basin scale wave is captured very well in the simulations. One noticeable difference is that the amplitudes of the solitary waves are larger in the numerical simulations and, at $x = 1.67$, there are more of them in the first group ($t = 50–75$ s). Another difference is that at $x = 1.67$ the interface in the numerical simulations is above that in the laboratory experiments between about $t = 55$ and $t = 140$ s. This may be in part due to the difference in how the interface height is computed: in the simulations the interface height is given by the zero density contour (centre of the pycnocline), in the laboratory experiments it is measured by an ultrasonic probe which measures the travel time of an acoustic pulse.

4. Summary

The numerical model simulates several aspects of the tilting tank experiments quite well. The overall energy decay is simulated very well however the thickening of the pycnocline is overestimated. In addition, the amplitudes of the solitary waves may be over-predicted in the model simulations, however this could be due in part to measurement differences. Further model tests are being done to see if the use of
additional terms (e.g., Rayleigh damping) in the momentum equations or a subgrid turbulence model (e.g., a Smagorinsky model) can lead to improvements.

Model runs with 600 grid points in the horizontal had significant non-monotonic BPE, indicating that the numerical error is too large. Simulations with 1200 grid points had almost monotonic BPE, however there were small oscillations during the first strongest breaking event.

References


