The degeneration of internal waves in lakes with sloping topography

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Abstract

Observations are presented from Lake Biwa (Japan) and Lake Kinneret (Israel) showing the ubiquitous and often periodic nature of high-frequency internal waves in large stratified lakes. In both lakes, high-frequency wave events were observed within two distinct categories: (1) Vertical mode one solitary waves with wavelength $\sim$100-500 m and frequency near $10^{-3}$ Hz and (2) sinusoidal vertical mode one waves with wavelength $\sim$5-30 m and frequency just below the local maximum buoyancy frequency near $10^{-2}$ Hz. The sinusoidal waves were associated with shear instability and were shown to dissipate their energy sporadically within the lake interior. Conversely, the solitary waves were found to be capable of propagating to the lake perimeter where they may break upon sloping topography, each releasing $\sim$1% of the total basin-scale internal wave energy to the benthic boundary layer.

Field observations in lakes, where the effects of the Earth’s rotation can be neglected, suggest that the basin-scale internal wave field may be decomposed into a standing seiche, a progressive nonlinear surge and a dispersive solitary wave packet. Laboratory experiments were used to quantify (1) the temporal energy distribution and flux between these three component internal wave modes and (2) the energy loss from the wave field as a response to high-frequency internal wave breaking at the boundary. Under moderate forcing conditions, the surge received up to 20% of the initial available potential energy during a nonlinear steepening phase and, in turn, conveyed this energy to the smaller-scale solitary waves as dispersion became significant. This temporal energy flux may be quantified in terms of the ratio of the linear and nonlinear terms in the nonlinear nondispersive wave equation. Wave breaking was observed to occur upon sloping topography, thus the ratio of the steepening timescale, required for the evolution of the solitary waves, to the traveltime of these waves controlled their development within the domain. The energy loss at the slope, the mixing efficiency and the breaker type were modelled in terms of the quiescent fluid properties and forcing dynamics, using an internal form of the Iribarren number.

These results are supported by comparison to field data and allow revision of a general model for the internal wave spectrum in lakes. The portion of the spectrum at the motions at the basin-scale and the near buoyancy frequency was found to be composed of progressive nonlinear waves. Strong and moderate forcing conditions were shown to excite solitary waves near $10^{-3}$ Hz, as described above, and sinusoidal nonlinear waves near $10^{-4}$ Hz, respectively. In both the field and laboratory the wave groups were observed to rapidly dissipate, thus suggesting that a periodically forced system with sloping topography will sustain a quasi-steady flux of 20% of the forcing energy to the benthic boundary layer at the depth of the metalimnion. It is recommended that this flux, which occurs through the nonhydrostatic and sub grid-scale nonlinear wave field now be parameterized into coupled hydrodynamic and water-quality models.
Dedication

To my parents.

“We are in the position of a little child entering a huge library, whose walls are covered to the ceiling with books in many different languages. The child knows that someone must have written those books. It does not know who or how. It does not understand the languages in which they are written. The child notes a definite plan in the arrangement of the books, a mysterious order, which it does not comprehend but only dimly suspects.”

–Albert Einstein
Contents

Abstract iii
Dedication v
List of Figures xi
List of Tables xiii
Acknowledgements xv
Preface xvii

1 Introduction 1

2 High-frequency internal waves 5
  2.1 Abstract .................................. 5
  2.2 Introduction ................................ 5
  2.3 Review of observational methods .................... 7
  2.4 Review of study areas .......................... 7
  2.5 Numerical and analytical methods ..................... 10
    2.5.1 Linear stability model ...................... 10
    2.5.2 Korteweg-de Vries model ..................... 12
  2.6 Observational results .......................... 15
    2.6.1 Ubiquitous nature .......................... 15
    2.6.2 Nonlinear steepening and boundary interaction .......... 16
    2.6.3 Phase-coherence ............................ 21
    2.6.4 Direct observation ........................ 26
  2.7 Linear stability results ........................ 29
    2.7.1 Water column microstructure profiles ............... 38
  2.8 Nonlinear wave model results ..................... 39
  2.9 Discussion .................................. 43
    2.9.1 Coherence and turbulence ...................... 43
    2.9.2 Growth rate versus decay rate .................. 44
    2.9.3 Thermocline trapping ........................ 45
    2.9.4 Eddy coefficients .......................... 46
    2.9.5 Energy flux paths .......................... 47
3 Internal wave energetics

3.1 Abstract .................................. 51
3.2 Introduction .............................. 51
3.3 Theoretical background ................. 52
3.4 Laboratory experiments .................. 58
  3.4.1 Experimental methods .................. 58
  3.4.2 Experimental observations .......... 59
  3.4.3 Decomposition of the internal wave field .. 62
  3.4.4 Experimental results .................. 67
3.5 Discussion ................................ 68
  3.5.1 Estimation of viscous damping .......... 68
  3.5.2 The progressive surge .................. 69
  3.5.3 Field observations .................... 72
3.6 Conclusions ............................. 74

4 Experiments on shoaling internal waves

4.1 Abstract .................................. 77
4.2 Introduction .............................. 77
4.3 Theoretical background ................. 80
4.4 Experimental methods .................... 85
4.5 Results .................................. 87
  4.5.1 Flow field ............................ 87
  4.5.2 Internal solitary wave energetics ........ 89
  4.5.3 Breaker observations ................. 93
  4.5.4 Breaker classification and reflection coefficient .... 95
  4.5.5 The breaking point .................... 100
4.6 Field observations ....................... 104
  4.6.1 Lake Pusiano .......................... 104
  4.6.2 An interpretation of the wave-spectrum .......... 106
4.7 Conclusions ............................. 111

5 Conclusions ............................... 113

A Two-dimensional internal wave equations

A.1 Abstract .................................. 117
A.2 Fundamental equations .................... 117
A.3 The Taylor-Goldstein equation ........................................ 119
A.4 The linear wave equation ............................................. 121
A.5 The Korteweg-de Vries equation .................................... 123
  A.5.1 The solitary wave solution ...................................... 127
  A.5.2 The hydrostatic approximation ................................. 128

B Linear wave equation initial value problem ......................... 131
  B.1 Abstract ......................................................... 131
  B.2 Solution of the initial value problem ............................ 131

Bibliography ..................................................................... 137
List of Figures

2.1 Map of lakes Kinneret and Biwa showing bathymetry and locations of relevant sampling stations ........................................... 8
2.2 Observations of wind, isotherm displacement and potential energy spectra in lakes Kinneret and Biwa .......................... 11
2.3 Observations of high-frequency internal waves in Lake Kinneret ................................................................. 17
2.4 Background temperature and velocity structure during the passage of a vertical mode two wave event ........................ 18
2.5 Time-averaged water column profiles before and after a mode two wave event in Lake Kinneret ................................. 19
2.6 Observations of high-frequency internal waves in Lake Biwa ................................................................. 20
2.7 Observations of a steepened internal Kelvin wave in Lake Kinneret ................................................................. 22
2.8 Background temperature and velocity structure during the passage of a steepened internal Kelvin wave ....................... 23
2.9 Observations of high-frequency internal waves in Lake Biwa ................................................................. 24
2.10 Time-averaged temperature and velocity structure during the passage of a high-frequency wave event in Lake Biwa ................................................................. 25
2.11 Spectra of integrated potential energy in Lake Biwa ................................................................. 26
2.12 Orthogonal phase, coherence and energy spectra for the high-frequency wave events ................................................................. 27
2.13 Detail showing vertical mode one solitary waves of depression progressing past thermistor chains ................................. 29
2.14 Isotherm and time averaged velocity and temperature profiles during the passage of high-frequency internal waves in Lake Kinneret ................................................................. 31
2.15 Plan view of horizontal velocity vectors during high-frequency wave events ................................................................. 33
2.16 Linear stability results for V2 wave events with $K = 0$ ................................................................. 34
2.17 Linear stability results for V1 wave events with $K = 0$ ................................................................. 35
2.18 Linear stability results for V1 wave events with $K \sim 10^{-4}$ m$^2$s$^{-1}$ ................................................................. 36
2.19 Linear stability results for V1 wave events with $K \sim 10^{-3}$ m$^2$s$^{-1}$ ................................................................. 37
2.20 Water column structure from PFP casts and unstable V2 modes ................................................................. 40
2.21 Water column structure from PFP casts and unstable V1 modes ................................................................. 41
2.22 Mean theoretical wavelength and mean nonlinear phase velocity of all stable modes ................................................................. 42
2.23 Maximum growth rate and decay rate of the most unstable mode ................................................................. 46
List of Tables

2.1 Details of PFP deployments and selected data .......................... 13
2.2 Characteristics of the unstable modes from the linear stability analysis 28
2.3 Vertical mode one solitary wave observations and KdV model results 30
2.4 Vertical mode two solitary wave observations and KdV model results 32

3.1 Summary of experimental runs ............................................. 61
3.2 Viscous damping equations .................................................. 69

4.1 Summary of experimental runs ............................................. 86
4.2 Tabulated list of field observations and calculated parameters .......... 108
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Preface

The thesis body is comprised of three succinct manuscripts (Chapters 2, 3 and 4) each containing an introduction, a review of the relevant literature and methodology and a presentation and discussion of the results. A general introduction (Chapter 1) and conclusion (Chapter 5) are used to motivate the overall study, outline the material to be presented and draw together the conclusions from each of the three manuscripts. The work contained in this thesis is wholly my own, although carried out under the supervision of Professors Ivey and Imberger.

CHAPTER 1

Introduction

When viewed from space, the Earth appears as a blue planet. Seventy to seventy-five percent of the Earth’s surface is covered with water. Although water is seemingly abundant, one must consider the amount of usable fresh water available. Only 0.61% of the Earth’s water is fresh and in liquid form at the surface 0.01% or underground 0.6% (Nace, 1967; U.S. Geological Survey, 1984). However, most ground water resources are not shallow enough to be tapped at an affordable cost. Annually, humans appropriate approximately 4% of the global fresh surface water resources (Postel et al., 1996) - equivalent to 20% of the total volume of the Laurentian Great Lakes. This demand for water is increasing with the global population, and although the total volume of fresh water on Earth remains constant, the quality and thus the amount of useable fresh water is effectively being reduced by pollution and contamination.

One of the main prognostic and diagnostic tools for assessing the quality of surface water, is coupled hydrodynamic and water-quality computational models. The predictive skill of these models is limited by (1) their ability to approximate the fundamental equations upon which they are based and (2) the accuracy with which the fundamental equations describe the processes occurring within the natural environment. It is becoming increasingly clear that of these processes, those which influence water-quality are not only the bio-chemical but are also the hydrodynamical fluid motions occurring within the waterbody (e.g. Mortimer, 1987).

Of fundamental importance to the hydrodynamics of lakes and reservoirs is the typically observed vertical gradient in temperature. The warm surface layer (the epilimnion) floats upon the colder main body (the hypolimnion). In these systems, the potential energy inherent in the thermal stratification stabilizes the water column and inhibits vertical transport and mixing; vertical gradients in pollutants, nutrients, plankton and oxygen result (e.g. Mortimer, 1974; Fischer et al., 1979; Imboden & Wüest, 1995; Imberger, 1998). Simple energy models and microstructure and tracer observations suggest that turbulent buoyancy flux, which erodes the ambient stratification, is negligible within the lake interior but occurs primarily at rates an order of magnitude greater than in the interior along the lake boundaries (e.g. Goudsmit et al., 1997; Imberger, 1998; Wüest et al., 2000; Saggio & Imberger, 2001). The physical processes responsible for this spatial heterogeneity of mixing are also believed to occur within larger scale oceanic flows. Observed vertical mixing rates within the vast ocean interior (e.g. Ledwell et al., 1993) are an order of
magnitude smaller than classical global estimates (Munk, 1966; Gregg, 1987). Observations show this difference to be accounted for by enhanced diapycnal mixing and dissipation within turbulent boundary layers upon sloping topography at the basin perimeter (e.g. Polzin et al., 1997; Ledwell et al., 2000). Within the littoral zone, this mixing drives enhanced local nutrient fluxes and bio-productivity (e.g. Sandstrom & Elliott, 1984; Ostrovsky et al., 1996; MacIntyre et al., 1999; Kunze et al., 2002).

Indirect observations suggest that the turbulent boundary layers are energized by internal wave activity. In lakes, internal waves are typically initiated by an external disturbance such as a surface wind stress. This stress advects surface water toward the lee-shore, thus displacing the internal strata along the length of the basin. Upon termination of the stress, the internal layers will oscillate in natural (and sometimes forced) modes about the equilibrium position. These internal wave modes maintain the turbulent boundary layers along the lake perimeter through (1) bed-shear induced by the currents of the basin-scale waves (Fischer et al., 1979; Fricker & Nepf, 2000; Gloor et al., 2000; Lemcchert et al., 2004) and (2) breaking of high-frequency internal waves upon sloping topography at the depth of the metalimnion (Thorpe et al., 1972; MacIntyre et al., 1999; Michallet & Ivey, 1999; Gloor et al., 2000). While the former process is relatively well understood, the latter remains comparatively unexplored.

Clearly, the skill of coupled hydrodynamic and water-quality models depends on their ability to simulate these fundamental physical processes. High-frequency internal waves are often nonhydrostatic and sub-grid scale; thus their evolution, propagation and breaking are not resolved by practical field scale computational models (e.g. Horn et al., 1999; Hodges et al., 2000; Boegman et al., 2001). Previous studies have identified the mechanisms by which a basin-scale internal wave will degenerate (Horn et al., 2001), including the production of high-frequency internal waves. However at present, the computational parameterization of these processes is not a viable option because the distribution and flux of energy between the various internal wave groups remains unknown (Imberger, 1998).

This thesis seeks to quantify the crucial energy transfer, in lakes, between the wind forced wave motions at the basin-scale and the small-scale turbulent mixing and dissipation due to wave breaking at the boundary. In Chapter 2, observations of high-frequency internal waves from the interior of two large stratified lakes are analyzed. The primary objectives are to determine physical characteristics of the high-frequency internal waves and to use this information to evaluate the role of
high-frequency internal waves as they propagate through the fluid and interact with the lake boundaries. Based upon the field results, a laboratory model is used in Chapter 3, to quantify the temporal energy distribution and flux between the various basin-scale internal wave modes and more importantly the downscale energy transfers to high-frequency internal waves capable of breaking upon sloping topography at the lake boundary. Chapter 4 seeks to quantify the energy loss from the breaking of the high-frequency internal waves and to cast these results in terms of parameters which are external to the evolving sub basin-scale flow (e.g. wind speed and direction, boundary slope, quiescent stratification, etc.). This will facilitate engineering application and parameterization into field-scale hydrodynamic models. In conclusion (Chapter 5), the results are summarized and placed within the context of what is presently known about the energetics and mixing within large stratified lakes.
High-frequency internal waves in large stratified lakes

2.1 Abstract

Observations are presented from Lake Biwa and Lake Kinneret showing the ubiquitous and often periodic nature of high-frequency internal waves in large stratified lakes. In both lakes, high-frequency internal wave events were observed within two distinct categories: (1) Solitary waves of the first vertical mode near a steepened Kelvin wave front and of the second vertical mode at the head of a thermocline jet. These waves were found to have wavelengths \( \sim 64–670 \) m and \( \sim 13–65 \) m, respectively, and were observed to excite a spectral energy peak near \( 10^{-3} \) Hz. (2) Sinusoidal vertical mode one waves on the crests of Kelvin waves (vertically coherent in both phase and frequency) and bordering the thermocline jets in the high shear region trailing the vertical mode two solitary waves (vertically incoherent in both phase and frequency). These waves were found to have wavelengths between 28–37 m and 9–35 m, respectively, and excited a spectral energy peak just below the local maximum buoyancy frequency near \( 10^{-2} \) Hz. The waves in (1) and (2) were reasonably described by nonlinear wave and linear stability models, respectively. Analysis of the energetics of these waves suggests that the waves associated with shear instability will rapidly dissipate their energy within the lake interior and are thus responsible for patchy turbulent events which have been observed within the metalimnion. Conversely, the finite amplitude solitary waves, which each contain as much as 1% of the basin-scale Kelvin wave energy, will propagate to the lake perimeter where they may shoal thus contributing to the maintenance of the benthic boundary layer.

2.2 Introduction

The degree of ambient stratification in lakes is significant in regulating the vertical transport of nutrients, plankton and oxygen. Accordingly, stratification may control the occurrence of hypolimnetic anoxia (e.g. Mortimer, 1987). Simple energy models (Imberger, 1998) and microstructure (e.g. Wüest et al., 2000; Saggio & Imberger, 2001) and tracer (e.g. Goudsmit et al., 1997) observations suggest that turbulent buoyancy flux, which erodes the ambient stratification, is negligible within the lake interior and occurs primarily at rates an order of magnitude greater than

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in the interior along the lake boundaries. The physical processes responsible for this spatial heterogeneity of mixing within the lacustrine environment are also believed to occur within larger scale oceanic flows. Observed vertical mixing rates within the vast ocean interior (e.g. Ledwell et al., 1993) are an order of magnitude smaller than classical global estimates (Munk, 1966; Gregg, 1987). Observations show this difference to be accounted for by enhanced mixing within topographic boundary layers (e.g. Polzin et al., 1997; Ledwell et al., 2000). Our present knowledge suggests that the augmented mixing and dissipation within the lacustrine benthic boundary layer results from shear driven turbulence as baroclinic currents oscillate along the lake bed and the breaking of high-frequency internal waves as they shoal upon sloping boundaries at the depth of the metalimnion (Imberger, 1998; Michallet & Ivey, 1999; Gloor et al., 2000; Horn et al., 2000). Interior mixing is believed to result from both shear and convective instability (Saggio & Imberger, 2001).

Observational records have shown high-frequency internal waves to be ubiquitous to lakes and oceans (see Garrett & Munk, 1975). Furthermore, high-resolution sampling (e.g. Saggio & Imberger, 1998; Antenucci & Imberger, 2001) has provided evidence that these high-frequency internal waves exist in narrow but discrete frequency bands approaching the buoyancy frequency, \( N \). Here \( N^2 = -\left( g/\rho_o \right) \left( \partial \rho / \partial z \right) \) where \( \rho(z) \) is the ambient density profile and \( \rho_o \) is a reference density. Both the narrowness of the observed frequency band and the existence of similar observations from a variety of lakes suggests common generation mechanisms. Several such mechanisms have been proposed: (1) shear instability (e.g. Woods, 1968; Thorpe, 1978; Sun et al., 1998; Antenucci & Imberger, 2001), (2) nonlinear steepening of basin-scale internal waves (e.g. Hunkins & Fliegel, 1973; Farmer, 1978; Mortimer & Horn, 1982; Horn et al., 2000), (3) internal hydraulic jumps (e.g. Apel et al., 1985; Holloway, 1987; Farmer & Armi, 1999), (4) excitation by intrusions and gravity currents (e.g. Hamblin, 1977; Maxworthy et al., 1998) and (5) flow interaction with boundaries (e.g. Thorpe et al., 1996; Thorpe, 1998). In this paper, the waves generated by these mechanisms are grouped into two fundamental classes: waves described by linear stability models (e.g. Kelvin-Helmholtz and Holmboe modes); and waves described by weakly nonlinear models (e.g. solitary waves).

In the following sections we analyze observations of high-frequency internal waves from the interior of two large stratified lakes. The primary objectives are to: (1) identify the occurrence of mechanisms which may lead to the excitation of high-frequency internal waves; (2) determine and compare the physical characteristics of the high-frequency internal waves (i.e. wavelength, phase velocity and spectral
2.3 Review of observational methods

The data presented has been extracted from that used in the studies of lakes Biwa and Kinneret by Antenucci et al. (2000) and Saggio & Imberger (1998, 2001). These articles contain a rigorous description of the sampling procedure and apparatus. Only a brief summary is provided here. The data were recorded in Lake Biwa using thermistor chains with a 15-s sampling interval. The thermistor chains were located near station BN50; deployed in a star-shaped array for 10 days during 1992 (figure 2.1c) and aligned roughly along a transect for 20 days in 1993 (figure 2.1d). In Lake Kinneret, thermistor chain data (figure 2.1a) were recorded for 18 days during 1997 at stations T2 (10-s intervals), T7 and T9 (120-s intervals), for 14 days during 1998 at station T3 (10-s intervals) and for 17 and 21 days during 1999 at stations T1 and T2 (10-s intervals), respectively. All thermistor chains were sampled with an accuracy of 0.01°C. Individual sensors were spaced at 1 m intervals within the metalimnion and up to 5 m intervals in the hypolimnion and epilimnion. The thermistor chain data were supplemented with microstructure data collected using a Portable Flux Profiler (PFP) equipped with temperature sensors (0.001°C resolution) and orthogonal two-component laser Doppler velocimeters (0.001 m s\(^{-1}\) resolution). Profiling vertically through the water column at a speed of approximately 0.1 m s\(^{-1}\) and a sampling frequency of a 100 Hz, the PFP resolved water column structure with vertical scales as small as 1 mm.

2.4 Review of study areas

Lake Kinneret (figure 2.1a) is approximately 22 km by 15 km, with a maximum depth of 42 m and an internal Rossby radius typically half the basin width.
Figure 2.1: (a) Lake Kinneret (33°N 36°E) bathymetry with locations of relevant sampling stations. Observations are presented from thermistor chains deployed at all stations during 1997 and at station T3 during 1998. (b) Lake Biwa (35°N 136°E) bathymetry with locations of relevant sampling stations. Thermistor chains were deployed near station BN50 during 1992 (c) and near stations BN50, 35m and 15.5m during 1993 (d). In panel d chains 1 and 4 are 130 m apart while chains 4 and 5 are 205 m apart.
June investigation of the basin-scale wave field (Antenucci et al., 2000) revealed the 24-h vertical mode one Kelvin (cyclonic) wave as the dominant response to the wind forcing (figure 2.2a-c). Vertical mode one, two and three Poincaré (anti-cyclonic) waves were also observed with periods of 12, 20 and 20 hours, respectively (figure 2.2a-c). High-frequency waves were observed by Antenucci & Imberger (2001) to occur in packets at locations where the crest of the 24-hr vertical mode one Kelvin wave is in phase with the lake’s intense diurnal wind forcing. The shear at the crest of the propagating Kelvin wave is thus augmented by wind shear at the base of the epilimnion. These waves were shown to energize a spectral energy peak just below the local $N$ near $10^{-2}$ Hz (figure 2.2c). An inviscid linear stability analysis demonstrated that unstable modes were possible; however, they did not clearly resolve the growth rate peaks in wavenumber space or rigorously compare the predicted unstable modes to the observed data. Therefore, they were unable to accurately determine the wavelength, direction of propagation and dissipation timescale of the observed high-frequency waves.

Lake Biwa (figure 2.1b) is approximately 64 km long with a maximum width of 20 km and minimum width of only 1.4 km. The main basin has a maximum depth 104 m and a typical internal Rossby radius of 5.4 km. An investigation of the September internal wave field (Saggio & Imberger, 1998) revealed vertical mode one and two basin-scale Kelvin waves with periods of 2 and 6 days, respectively, and vertical mode one basin-scale Poincaré waves with horizontal modes of 1 through 4 and periods between 12 and 24 h (figure 2.2d-f). High-frequency internal waves associated with internal undular bores and hydraulic jumps were observed near a steepened Kelvin wave after the passage of a typhoon. These large amplitude high-frequency waves were nonlinear in appearance and energized a spectral energy peak near $10^{-3}$ Hz (figure 2.2f). Further investigation of the Lake Biwa data set (Maxworthy et al., 1998) revealed vertical mode two internal solitary waves which were believed to result from the gravitational collapse of shear instabilities near the crests of the Kelvin waves. These studies on Lake Biwa were unable to demonstrate the occurrence of shear instability and, as for Lake Kinneret, determine the physical characteristics of the high-frequency wavefield. In particular, the wavelength, direction of propagation and dissipation timescale of the observed high-frequency waves. These issues for both field studies are addressed here.
2.5 Numerical and analytical methods

To interpret and analyze the high-frequency wave domain within the Lake Biwa and Lake Kinneret field records, linear stability and weakly nonlinear models were applied and rigorously compared to the observed field data sets.

2.5.1 Linear stability model The Taylor-Goldstein equation describes the growth and stability behavior of small perturbations in inviscid fluids with ambient shear and continuous stratification. The Miles-Howard condition stipulates that the sufficient condition for stability to small perturbations is that the local gradient Richardson number ($Ri$) be greater than one quarter everywhere in the flow. Here

$$Ri = N^2 / (\partial u / \partial z)^2$$

where $u = (u, v, w)$ is the local fluid velocity. Unstable modes may grow into finite amplitude perturbations within the flow (see Batchelor, 1967; Turner, 1973).

To determine whether the high-frequency internal wave modes observed in the field data may be modelled by linear shear instability we used a method similar to that of Sun et al. (1998). However, we retained the viscous and diffusive terms in the governing equations to allow for preferential damping of large wavenumber instabilities (Smyth et al., 1988). Our three-dimensional flow geometry was first simplified into two-dimensional horizontal velocity profiles $U(z)$ decomposed from the zonal $k$ and meridional $l$ components along 32 horizontal radii which are oriented at $11.25^\circ$ increments. We then recovered the directional nature of the instabilities through combination of the unique solution along each axis in wavenumber space.

We acknowledge that decomposition of a three-dimensional flow field into multiple two-dimensional solution planes precludes three-dimensional primary instabilities. However, these instabilities are not expected in geophysical flows as they have been shown to be restricted to a small region of parameter space where the Reynolds number $Re$ is less than 300 (Smyth & Peltier, 1990). Here, $Re = (U\delta)/\nu$, where $U$ is a variation in velocity over a length scale $\delta$ and $\nu$ is the kinematic velocity of the fluid.

We numerically determined the stability of the water-column along each axis to forms of the equations of motion subject to infinitesimal wave-like perturbations of the vertical velocity field within a Boussinesq and hydrostatic flow of the form

$$\psi(x, z, t) = \Re \{\hat{\psi}(z) \exp[i\kappa(x - ct)]\}$$

(2.1)

where $\kappa = \sqrt{k^2 + l^2}$ is the horizontal wavenumber, $\hat{\psi}(z) = \hat{\psi}_r + i\hat{\psi}_i$ is the complex wavefunction, $c = c_r + ic_i$ is the complex phase speed and $\omega_i = \kappa c_i > 0$ represents...
Figure 2.2: Observations of wind and isotherm displacement in Lake Kinneret (1998) and Lake Biwa (1993) after Antenucci et al. (2000) and Saggio & Imberger (1998), respectively. (a) Wind speed at T3 corrected to 10 m above the water surface; (b) isotherms at 2°C intervals calculated through linear interpolation of thermistor chain data at T3; (c) power spectra of integrated potential energy (Antenucci et al., 2000) from panel b; (d) wind speed at chain 5 (BN50) corrected to 10 m above the water surface; (e) isotherms at 2°C intervals calculated through linear interpolation of thermistor chain data at chain 5; (f) power spectra of integrated potential energy from panel e. Data in panels a, b, d and e have been low-pass filtered at 1 h. The bottom isotherms in panels b and e are 17°C and 10°C, respectively. In panels c and f, $N$ is approximately $10^{-2}$ Hz. Spectra have been smoothed in the frequency domain to improve confidence with confidence at the 95% level shown by the dotted lines.
the growth rate of an unstable perturbation.

Mean flow profiles of $N(z)$ and $U(z)$ were obtained by depth averaging observed water column profiles obtained over a finite time interval $\Delta t$ (Table 2.1). Sensitivity analysis revealed that depth and isopycnal averaged profiles were insignificantly variant as a result of smoothing associated with transforming isopycnal averaged quantities to the vertical depth coordinate of our stability model. Furthermore, isopycnals are nearly horizontal since $\Delta t$ is much less that the basin scale wave period. To reduce the computational demand the time averaged profiles were averaged into vertical bins ranging between 5 and 15 cm (Table 2.1). A matrix eigenvalue method modified after Hogg et al. (2001) was used to solve the sixth order viscous stability equation. As opposed to the methods used by Antenucci & Imberger (2001) or Hogg et al. (2001), in our development we have not assumed inviscid and nondiffusive fluid or made the long wave assumption, respectively. This stability equation is, therefore, a generalization of the Taylor-Goldstein equation that includes viscosity and diffusivity (Koppel, 1964)

\[
-c^2[\partial^2 - \kappa^2]\hat{\psi} + c[2iK_c(\partial^2 - \kappa^2) - U_{zz}\hat{\psi} + [K_c^2(\partial^2 - \kappa^2) - U_{zzz}K_cK_c\partial(\partial^2 - \kappa^2) - U^2(\partial^2 - \kappa^2)] + 2iU_{z}\hat{\psi}] = 0 \quad (2.2)
\]

where $K_c = K/\kappa$. The $z$ subscript (on $U$) and $\partial$ are used to represent the ordinary and partial derivative with respect to $z$, respectively. Here $K$ has been defined an ‘eddy viscosity’ or ‘eddy diffusivity’ parameterization and a Prandtl number of unity has been assumed.

For various $N$ and $U$ profiles and also for given $K_c$ are the values of $\omega_i$, $\hat{\psi}$ and $c$ computed (over a range of $\kappa$). Vertical mode one through three solutions were evaluated for wavelengths at 2 m intervals between 1 and 149 m. Along each axis only the solutions at each wavenumber with the highest growth rate were kept for further analysis. Our results, which involve the determination of the stability of a time-variant flow over a finite time-averaged interval, may be considered exact for $\omega_i \gg 1/\Delta t$ (Smyth & Peltier, 1994).

### 2.5.2 Korteweg-de Vries model

Internal solitary waves may be modelled to first order by the weakly nonlinear KdV equation

\[
\eta_t + c_0\eta_x + \alpha \eta \eta_x + \beta \eta_{xxx} = 0 \quad (2.3)
\]
Table 2.1: Details of PFP deployments and selected data from casts averaged over the time intervals \( \Delta t \). Overbar denotes depth averaged quantity. The last two rows contain temporally averaged frequency characteristics of the high-frequency wave events observed using thermistor chains at the indicated locations.

where \( \zeta(x, z, t) \approx \eta(x, t)\hat{\psi}(z) \) is the wave amplitude, \( c_o \) is the linear long-wave speed and subscripts denote differentiation. A balance between nonlinear steepening \( \eta_{xx} \) and dispersion \( \eta_{xxx} \) results in waves of permanent form.

The coefficients \( \alpha \) and \( \beta \) are known in terms of the water column properties \( \rho(z) \) and \( U(z) \) and the modal function \( \hat{\psi}(z) \) (Benney, 1966)

\[
\alpha = \frac{3}{2} \int_0^H \rho(c_o - U)^2 \hat{\psi}^3 dz \quad \beta = \frac{H}{2} \int_0^H \rho(c_o - U)^2 \hat{\psi}^2 dz
\]

(2.4)

where \( H \) is the height of the water column and \( \hat{\psi} \) is determined from numerical solution of (2.2) with \( \kappa = 0 \). Note that in setting \( \kappa = 0 \) it was also necessary to specify \( K = 0 \) in order to maintain a finite \( K_c \), otherwise equation (2.2) will represent modes in an infinitely viscous fluid.

From (2.3) the solitary wave solution is (Benney, 1966)

\[
\eta(x - c_1t) = a \text{sech}^2 \left( \frac{x - c_1t}{L} \right)
\]

(2.5)

where the phase velocity \( c_1 \) and horizontal length scale \( L \) are given by

\[
c_1 = c_o + \frac{1}{3} \alpha a \\
L = \sqrt{\frac{12\beta}{a\alpha}}
\]

(2.6)
and \( a \) is the solitary wave amplitude. Considering the \( \text{sech}^2 \) solitary wave shape, Holloway (1987) suggests that the wavelength \( \lambda \approx 3.6L \).

To apply the continuous KdV theory described above to field observations of finite amplitude internal waves requires a knowledge of the velocity and density structure of the water column. In practice, this requires an a priori knowledge of the spatial and temporal distribution of the waves to be sampled. While this may not be a hindrance in large scale and periodic tidal flows (e.g. Apel et al., 1985), in lakes these waves are of much smaller scale and do not exhibit a surface signature which is visible through non-invasive remote sensing (e.g. satellite) techniques. Historically, such limitations on the quality of field data has prevented comparison of the continuous KdV theory described above to field observations of internal solitary waves in lakes. Qualitative or simplified models which neglect mean shear and employ layered approximations of the continuous density profile have been applied (e.g. Hunkins & Fliegel, 1973; Farmer, 1978).

Vertical mode one internal solitary waves can be analytically modelled in a two-layer Boussinesq and hydrostatic flow with no mean shear and of depth \( h_1 \) and density \( \rho_1 \) over depth \( h_2 \) and density \( \rho_2 \) using equation (2.6) with

\[
\alpha = \frac{3c_o}{2h_1 h_2} (h_1 - h_2) \quad \beta = \frac{c_o h_1 h_2}{6}
\]

where \( c_o = \sqrt{g'(h_1 h_2)/(h_1 + h_2)} \).

Vertical mode two internal solitary waves may be modelled using the three layer theoretical solution for wave celerity by Schmidt & Spigel (2000). In this model the mode two phase speed \( c_2 \) is

\[
c_2 = c_{o2} \frac{4}{3} \sqrt{\frac{2a}{h_2}} \quad c_{o2} = \sqrt{\frac{gh_2}{4} \frac{(\rho_1 - \rho_3)}{\rho_3 + \rho_1}}
\]

where \( c_{o2} \) is the speed of an infinitesimal mode one wave, the upper or lower interface wave amplitude is \( a \) and the ambient fluid is characterized by a middle layer of thickness \( h_2 \) and upper and lower layers of density \( \rho_1 \) and \( \rho_3 \), respectively. The half wavelength between \( a \) and \( a/2 \) is determined empirically as

\[
\lambda_{1/2} = 1.98a + 0.48h_2
\]

In this study we apply the approximate layered models using observations from thermistor chains. Where possible we supplement these results by applying the continuous KdV model to observations from nearby PFP casts.
2.6 Observational results

2.6.1 Ubiquitous nature

Lake Kinneret (27 to 30 June 1998) – Isotherms at station T3 (figure 2.3b) revealed basin-scale wave activity that was in phase with the surface wind forcing (figure 2.3a). Crests of the basin-scale 24-h vertical mode one Kelvin wave were observed near days 182.7, 183.7, 184.7 and 185.7, while crests of the 12-h vertical mode one basin-scale Poincaré wave were observed at the 24-h Kelvin wave crests and troughs. Relatively high-frequency and small amplitude vertical mode one waves, vertically coherent in both phase and frequency (figure 2.3c,d), were apparent riding on the Kelvin wave crests during the periods of intense surface wind forcing. These waves are hypothesized to result from shear instability at the base of the epilimnion (Antenucci & Imberger, 2001).

There is also evidence of high-frequency vertical mode two waves in this record. The interaction of the 24-h Kelvin wave trough, the 12-h Poincaré wave crest and the vertical mode two 20-h Poincaré wave was observed to cause a periodic constriction of the metalimnion. Immediately following this constriction and prior to the subsequent Kelvin wave crest, a packet of vertical mode two large amplitude internal solitary waves with an irregular lower amplitude wave tail was observed (figure 2.3e,f). Unlike the waves of figure 2.3c,d these waves vary vertically in frequency and phase and are followed by an abrupt splitting of the metalimnion. Analysis of PFP casts (recorded at the times denoted by arrows in figure 2.4a,b) show an overlay of the background velocity field and the isotherm displacements for the wave event in figure 2.3e. This event is visually consistent with an intrusive current or a metalimnion jet driven by the local vertical mode two compression or the collapse of a mixed region. The area of high shear above and below the jet is bounded by the irregular wave tail and regions where $Ri < 1/4$ (figure 2.5f). This suggests the possibility of localized shear instability. Some simple calculations may be used to determine if the mode-two waves and horizontal jet indeed result from the collapse of a mixed region - the mechanism proposed for Lake Biwa by Maxworthy et al. (1998). From figure 2.4a we calculate a radially spreading mixed fluid volume of $\pi [(0.1 \text{ m s}^{-1})(13000 \text{ s})]^2 (5 \text{ m}) \approx 2.7 \times 10^7 \text{ m}^3$; this is 0.2 % of the lake volume. Assuming a 20% mixing efficiency (Ivey & Imberger, 1991), $4 \times 10^8 \text{ J}$ of energy are needed to mix this fluid from equal parts of the epilimnion and hypolimnion. This is approximately 4% of the total energy in the internal Kelvin wave field, estimated as $5 \times 10^{10} \text{ J}$ (see Imberger, 1998), which is a reasonable value. However, if this mixed fluid did originate from the Kelvin wave crest which passed station T3 3.6 hours
prior, where has it been in the interim?

Lake Biwa (11 to 15 September 1992) – Isotherms from thermistor chain 5 (figure 2.6b) revealed a period of metalimnion compression, beginning gradually on day 255 and extending until day 258. Saggio & Imberger (1998) described this event as an increase in the buoyancy frequency to a maximum of 0.022 Hz and associated it's presence with the passage of a six-day basin-scale vertical mode two Kelvin wave. Throughout this metalimnion compression, the two-day vertical mode one basin-scale Kelvin wave is evident with crests near days 254.5, 256.5 and 258.5. High-frequency vertical mode one waves, similar to those in Lake Kinneret, were observed on the Kelvin wave crest at day 256.6 (figure 2.6d) and during moderately strong winds as the metalimnion expanded (days 257.7 to 258.3). Vertical mode two wave events, again remarkably similar to those in Lake Kinneret, were also observed throughout the metalimnion compression; for example on days 255.2, 256.3 (figure 2.6c) and 258.0.

2.6.2 Nonlinear steepening and boundary interaction

Lake Kinneret (1 to 4 July 1997) – The Kelvin wave crests at station T2 were in phase with the diurnal wind forcing on days 177.7, 178.7, 179.7 and 180.7 (figure 2.7a,b). At station T9 (figure 2.7c) there was a 12-h phase lag with the Kelvin wave crests arriving on days 178.2, 179.2, 180.2 and 181.2. Comparison of figures 2.7b and 2.7c over this time interval shows the leading edge of each Kelvin wave trough to have steepened; each wave changing from a sinusoidal shape to one exhibiting a gradual rise in isotherm depth followed by an abrupt descent after the passage of the wave crest. The trough regions showed evidence of high-frequency wave activity and metalimnion expansion, but the 2-min sampling interval did not reveal internal solitary waves or other high-frequency waves. Progressing further in a counter-clockwise direction, the Kelvin wave crests at station T7 were observed to, once again, possess a sinusoidal shape (figure 2.7d); however, uniformly distributed relatively high-frequency isotherm oscillations were also observed. These observations suggest a steepening mechanism similar to that observed in long narrow lakes. This nonlinear steepening may be influenced by the nonuniform bathymetry/topography of Lake Kinneret near T9 (cf. Farmer, 1978; Mortimer & Horn, 1982; Horn et al., 2000).

Data from PFP casts (recorded at the times denoted by arrows in figure 2.8a,b) were linearly interpolated to show an overlay of the background velocity field and the isotherm displacements for the steepened wave front shown as a shaded region in figure 2.7c. Current speeds between 30 and 35 cm s$^{-1}$ were observed as the Kelvin
Figure 2.3: Observations from station T3 in Lake Kinneret 1998: (a) Ten-min average wind speed corrected from 1.5 to 10 m, (b) 2°C isotherms for a 4-d observation period, (c) magnified view of shaded region c in (b) showing 1°C isotherms, (d) magnified view of shaded region d in panel b showing 1°C isotherms, (e) magnified view of shaded region e in panel b showing 1°C isotherms, (f) magnified view of shaded region f in panel b showing 1°C isotherms. Wind and temperature data were collected at 10-s intervals, with isotherms calculated through linear interpolation. The bottom isotherm in panel b is 17°C.
Figure 2.4: PFP observations near station T3 in Lake Kinneret 1998 of background temperature and velocity structure during the passage of a vertical mode 2 wave event depicted in figure 2.3e on day 184: (a) isotherms from T3 data collected at 10-s intervals, superposed on contours of current velocity in north-south direction (north positive) derived from PFP casts as indicated by arrows and averaged into 15-cm vertical bins; (b) isotherms as in panel a, superposed on contours of current velocity in east-west direction (east positive) derived from PFP casts as in panel a.
2.6. Observational results

Figure 2.5: Time-averaged water column profiles from PFP casts depicted in figure 2.4a,b: before the mode two wave event on day 184.36 (a) temperature $T$, (b) $N^2$, (c) $Ri$; and after the mode two wave event on day 184.36 (d) $T$, (e) $N^2$, (f) $Ri$. Time-averaged water column profiles from PFP casts depicted in figure 2.8a,b: before the steepened wave front on day 180.33 (g) $T$, (h) $N^2$, (i) $Ri$; and after the steepened wave front on day 180.33 (j) $T$, (k) $N^2$, (l) $Ri$.

wave crest constricted the local epilimnion thickness (figure 2.8a). These speeds were reduced to 15 to 20 cm s$^{-1}$ with the passage of the 5 m descending front. Hypolimnetic current speeds were generally below 10 cm s$^{-1}$. The flow was baroclinic in nature, with velocities 180° out of phase across the Kelvin wave crest (figure 2.8b). Within the epilimnion, there was a counter-clockwise rotation in current direction from NNW through SSW as the Kelvin wave passed. These observations are consistent with the theoretical model of the passage of a Kelvin wave in an idealized three-layer basin matching the dimensions of Lake Kinneret (Antenucci et al., 2000). Averaging data from PFP casts, both before and after the front (figures 2.5g-l), revealed the local $Ri$ to be below 1/4 at the upper and lower bounds of the high $N^2$ region which indicated the possibility of shear instability both above
Figure 2.6: Observations from chain 5 (BN50) in Lake Biwa 1992: (a) wind speed measured at 1.5 m, low pass filtered 10-min intervals (b) 2°C isotherms for a four day observation period, (c) magnified view of shaded region c in (b) showing 1°C isotherms, (d) magnified view of shaded region d in (b) showing 1°C isotherms. Wind and temperature data were collected at 15-s intervals, with isotherms calculated through linear interpolation. The bottom isotherm in (b) is 10°C.
2.6. Observational results

Lake Biwa (3 to 7 September 1993) – Observations from thermistor chain 5 showed crests associated with the two day vertical mode one basin-scale Kelvin wave on days 247.8 and 249.8 (figure 2.9b). The first Kelvin wave, observed at the thermistor chain approximately 24 h after the passage of a large typhoon (figure 2.9a), appeared as a gradual increase in the level of the isotherms followed by a rapid descent and a subsequent splitting of the metalimnion. This Kelvin wave is similar in appearance to the steepened Kelvin waves observed in Lake Kinneret at station T9 (figure 2.7c). A magnified view of the abrupt wave front (figure 2.9c) revealed a series of internal steps of depression each followed by a packet of large amplitude vertically coherent internal solitary waves and an irregular train of lower amplitude waves. These waves manifest a broad spectral energy peak near $10^{-3}$ Hz [figure 2.2 and Saggio & Imberger (1998)] and are identical in character to 15-s observations of a steepened longitudinal seiche in Seneca Lake (Hunkins & Fliegel, 1973, their figure 9). It remains unclear whether these high-frequency waves are generated through nonlinear steepening of basin-scale internal waves (cf. Horn et al., 2000) or boundary interaction as suggested by Saggio & Imberger (1998).

Data averaged from PFP casts both before and after the passage of the abrupt front (recorded at the times denoted by arrows on figure 2.9b) were used to determine the temporal variation of the background stratification and flow velocity (figure 2.10). Before the front, strong shear was observed across the thermocline at a depth of 10 m (figure 2.10a-c). Above this region in the epilimnion the $R_i$ is below $1/4$ (figure 2.10d) which indicates that the high-frequency waves observed through this high shear region (figure 2.9e) may result from shear instability. $R_i$ values below $1/4$ were also observed through the hypolimnion and benthic boundary layer due to the homogeneous nature of the water column. Once again the passage of the front reduces the intensity of the stratification, abruptly lowering the thermocline depth to approximately 20 m (figure 2.10f,g). Strong shear and $R_i < 1/4$ were also observed at this depth (figure 2.10e,h).

High-frequency wave events similar to those in figure 2.3c-f and figure 2.6c,d were observed throughout the four day observation period. For example, vertical mode one sinusoidal waves (figure 2.9d) occurred during the intense wind forcing on day 247.3 and vertical mode two internal solitary waves preceded an abrupt expansion of the metalimnion and irregular wave tail on day 247.6 (figure 2.9e).

2.6.3 Phase-coherence The close proximity and precise spacing of the thermistor chains deployed in the star shaped array in Lake Biwa during 1992 (fig-
Figure 2.7: Observations from Lake Kinneret 1997: (a) Ten-min average wind speed corrected from 1.5 to 10 m; (b) station T2, (c) station T9 and (d) station T7 1°C isotherms for a four day observation period. Wind and T2 data were collected at 10-sec intervals; however, T2 data were decimated to 120-s intervals to match the frequency of data collection at T7 and T9.
Figure 2.8: PFP observations near station T9 in Lake Kinneret 1997 of background temperature and velocity structure during the passage of the steepened wave front depicted in the shaded region of figure 2.7c on day 180: (a) isotherms from T9 data collected at 120-sec intervals, superposed on contours of current speed derived from PFP casts as indicated by arrows and averaged into 15 cm vertical bins; (b) isotherms as in panel a, superposed on contours of current azimuth direction derived from PFP casts as in panel a.
Figure 2.9: Observations from chain 5 (BN50) in Lake Biwa 1993: (a) Wind speed collected at Koshinkyoku Tower, 1 km north of BN50, resampled at 10-min intervals, (b) 2°C isotherms for a four day observation period, (c) magnified view of shaded region c in (b) showing 2°C isotherms, (d) magnified view of shaded region d in (b) showing 1°C isotherms, (e) magnified view of shaded region e in (b) showing 1°C isotherms. Temperature data were collected at 15-s intervals, with isotherms calculated through linear interpolation. The bottom isotherm in (b) and (c) is 10°C. Arrows in (b) denote PFP profiles taken at BN50 on days 247.7194, 247.7292, 247.7375, 248.4736, 248.4951, 248.5493, 248.5597, 248.5681, 248.5771.
Figure 2.10: Background temperature and velocity structure near BN50 in Lake Biwa 1993 as averaged from PFP profiles denoted by arrows in figure 2.9b: (a) North-south (solid line) and east-west (dotted line) velocity profiles on day 247 (velocities are positive to the north and east), (b) temperature, $T$, profile on day 247, (c) buoyancy frequency squared on day 247, (d) gradient Richardson number on day 248, (e) north-south (solid line) and east-west (dotted line) velocity profiles on day 248, (f) temperature profile on day 248, (g) buoyancy frequency squared on day 248, (h) gradient Richardson number on day 248.

Spectral density curves for these high-frequency wave events at chain 5 reveal energy peaks near $10^{-2}$ Hz (figure 2.11). Phase-coherence between chains 3 and 2 and chains 1 and 4 show high coherence at the frequencies of the spectral density peaks $A$, $B$ and $C$ (figure 2.12). The 6-m spacing between chain pairs and the temporal phase (figure 2.12) was used to determine the orthogonal components of spectral velocity at each of the spectral density peaks. Division of the spectral velocity com-
2.6.4 Direct observation Internal solitary waves are not sinusoidal in character and are thus not amenable to phase-coherence analysis. However, their physical characteristics may be determined through direct observation since only one positive progressive wave will match the observations at three or more non-collinear thermistor chains.

We considered three large vertical mode one waves of depression ordered according to wave amplitude on the steepened Kelvin wave front between days 247.95 and 248 (figure 2.9c) and separated the flow at thermistor chains 1, 4 and 5 (figure 2.1d) into two contiguous layers divided by the 20°C isotherm (figure 2.13). Direct observation gives an average crest to crest period of approximately 650 s or a mean...
Figure 2.12: Orthogonal phase, coherence and energy spectra for the high-frequency wave events identified in figure 2.11. Shading indicates frequency of peaks A, B and C as labelled. (a), (c) and (e) depict north-south components, while (b), (d) and (f) depict east-west components. Temporal phase is positive to the north and east.
Table 2.2: Characteristics of the unstable modes from the linear stability analysis results presented in figures 2.16 to 2.19. SD denotes standard deviation. Temporally averaged characteristics of the high-frequency waves shown in figures 2.6c,d and at the spectral peaks in figure 2.11 as determined using phase-coherence correlations across the star shaped thermistor chain array deployed in Lake Biwa during 1992 (figure 2.1c).

### Most unstable mode

<table>
<thead>
<tr>
<th>Location</th>
<th>Event</th>
<th>$K$ (m$^2$s$^{-1}$)</th>
<th>Wavelength (m)</th>
<th>Celerity (cm$s^{-1}$)</th>
<th>Period (s)</th>
<th>Observed period (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T1</td>
<td>V1</td>
<td>0</td>
<td>5</td>
<td>20</td>
<td>24</td>
<td>263</td>
</tr>
<tr>
<td>T2</td>
<td>V1</td>
<td>0</td>
<td>9</td>
<td>18</td>
<td>51</td>
<td>256</td>
</tr>
<tr>
<td>T3</td>
<td>V2</td>
<td>0</td>
<td>8</td>
<td>6</td>
<td>133</td>
<td>110</td>
</tr>
<tr>
<td>BN50</td>
<td>V2</td>
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<td>9</td>
<td>300</td>
<td>116</td>
</tr>
<tr>
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<td>15</td>
<td>847</td>
<td>263</td>
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<tr>
<td>T2</td>
<td>V1</td>
<td>$10^{-3}$</td>
<td>119</td>
<td>14</td>
<td>850</td>
<td>256</td>
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Mean±SD of the most unstable modes beneath the observed spectral energy peaks

<table>
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<th>$K$ (m$^2$s$^{-1}$)</th>
<th>Wavelength (m)</th>
<th>Celerity (cm$s^{-1}$)</th>
<th>Period (s)</th>
<th>Observed period (s)</th>
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<td>T1</td>
<td>V1</td>
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<td>BN50</td>
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<td>33±12</td>
<td>10±1</td>
<td>340±114</td>
<td>116</td>
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</table>

Phase-coherence observations

<table>
<thead>
<tr>
<th>Peak A</th>
<th>Event</th>
<th>Wavelength (m)</th>
<th>Celerity (cm$s^{-1}$)</th>
<th>Observed period (s)</th>
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</thead>
<tbody>
<tr>
<td>Peak A</td>
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</tr>
<tr>
<td>Peak B</td>
<td>V1</td>
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<td>-</td>
</tr>
<tr>
<td>Peak C</td>
<td>V2</td>
<td>35</td>
<td>16</td>
<td>-</td>
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</tbody>
</table>

temporal frequency $f_t = 1.5 \times 10^{-3}$ Hz. Note that for internal solitary waves which are non-sinusoidal and irregularly spaced the temporal frequency is distinct from the spatial frequency, $f_s = c/\lambda$. Analysis of figures 2.13 and 2.1d allowed determination of the average wavelength $\sim 670$ m and phase speed $\sim 180$ cm$s^{-1}$ (Table 2.3). However, visual estimates of the relative locations of thermistor chains 1, 4 and 5 are 50% lower than the the locations determined via non-differential GPS. This indicates that the wavelength and phase speed may actually be of the order 300 m and 100 cm$s^{-1}$, respectively.

We also considered the vertical mode two internal solitary waves shown in figure 2.6c. Dividing the flow at the 14°C and 24°C isotherms into three contiguous layers (not shown) the wavelength ($\sim 65$ m) and phase speed ($\sim 46$ cm$s^{-1}$) of the leading vertical mode two internal solitary wave in figure 2.6c was determined by direct
2.7 Linear stability results

Figure 2.13: Detail of figure 2.9c showing vertical mode one solitary waves of depression progressing past thermistor chains 1, 4 and 5 (figure 2.1d).

The observations presented above suggest two classes of high-frequency wave events suitable for linear stability analysis: (1) the vertically coherent mode one waves as observed on the crest of a Kelvin wave, henceforth referred to as V1 events and (2) the vertically incoherent mode one waves bounding the jet-like metalimnetic currents which follow vertical mode two solitary waves, henceforth referred to as V2 events. To perform a linear stability analysis on events of this type we require sufficient consecutive PFP casts to establish mean profiles of background density and horizontal velocity while the waves are observed at a particular thermistor chain. The linear stability model was thus applied to the V2 wave events at T3 and BN50 (figures 2.4a,b, 2.5d-f and 2.9e, 2.10a-d, respectively) and V1 wave events at T1 and T2 during 1999 in Lake Kinneret (figure 2.14). The lack of PFP casts during the V1 wave events presented thus far has required recourse to 1999 observations. During the passage of these V1 wave events the water column at T1 and T2 (figure 2.14) exhibited a strong advective horizontal velocity in the epilimnion to a depth of 8 m, a thermocline near 15 m in depth and $Ri$ below $1/4$ within the epilimnion and
Table 2.3: Comparison of observations and a weakly nonlinear model for the large amplitude vertical mode one internal waves shown in figures 2.9c, 2.10a-d, 2.13 and 2.22a,b. \( \Theta \) denotes temperature, see text for description of other symbols. The \( \dagger \) and \( \ddagger \) symbols denote values calculated using \( T_s = \lambda/c \) and \( \lambda = T_s c \), respectively, where the quantities have been obtained as given in this table.

<table>
<thead>
<tr>
<th>Model</th>
<th>Continuous density &amp; shear</th>
<th>Two-layer KdV</th>
<th>Direct observation</th>
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<tr>
<td>( h_2 ) (m)</td>
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<tr>
<td>( \Theta_1 ) (°C)</td>
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<td>-</td>
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<td>( \Theta_2 ) (°C)</td>
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<tr>
<td>( \rho_1 ) (kg m(^{-3}))</td>
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<td>( \rho_2 ) (kg m(^{-3}))</td>
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<td>-11.4</td>
<td>-11.4</td>
<td>-11.4</td>
</tr>
<tr>
<td>( L ) (m)</td>
<td>18±3</td>
<td>30</td>
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<tr>
<td>( \lambda ) (m)</td>
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<td>108</td>
<td>( \sim 335 - 670 \dagger )</td>
</tr>
<tr>
<td>( c_o ) (cm s(^{-1}))</td>
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<td>-</td>
</tr>
<tr>
<td>( c ) (cm s(^{-1}))</td>
<td>56±4</td>
<td>69</td>
<td>( \sim 90 - 180 )</td>
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<td>( T_s ) (s)</td>
<td>114( \dagger )</td>
<td>157( \dagger )</td>
<td>373±16</td>
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<tr>
<td>( f_t ) (Hz)</td>
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<td>-</td>
<td>( 1.5 \times 10^{-3} )</td>
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</tbody>
</table>

benthic boundary layer. For a rigorous description of the mean flow characteristics observed during the passage of the high-frequency waves depicted in figure 2.14 the reader is referred to Antenucci & Imberger (2001). Figure 2.15 shows a plan view of the horizontal velocity vector field for each of the V1 and V2 events.

Results are presented in figures 2.16 through 2.19 for linear stability solutions of the V1 and V2 events with \( 0 \leq K \leq 10^{-3} \) m\(^2\) s\(^{-1}\). For each event, the top panels a and d show the maximum growth rate in wavenumber space, the right panels b and e compare the frequency domain of the maximum growth rate at each wavenumber to the observed isotherm displacement spectra, and the bottom panels c and f show the growth rate at each wavelength. Instabilities within the shaded region of panels b and e in each figure have growth periods which are greater than the period over which the background flow is averaged from PFP casts (ie. \( (\kappa c_i)^{-1} > \Delta t \) and thus solutions within these regions must be interpreted with caution. These solutions are valid if the shearing stress is maintained longer than \( (\kappa c_i)^{-1} \). In the present case this is likely as we are concerned with maximum growth periods \( \sim 60 \) min (figure
2.7. Linear stability results

Figure 2.14: Observations from Lake Kinneret 1999: (a) Station T1 1°C isotherms, (b) station T2 1°C isotherms. In both (a) and (b) data were collected at 10-s intervals, while the dotted vertical lines denote PFP casts. (c) North-south (solid line) and east-west (dotted line) velocity profiles from PFP casts at T1 (velocities are positive to the north and east), (d) temperature, $T$, profile (dotted line) and buoyancy frequency squared (solid line) from PFP casts at T1, (e) gradient Richardson number from PFP casts at T1, (f) north-south (solid line) and east-west (dotted line) velocity profiles from PFP casts at T2 (velocities are positive to the north and east), (g) temperature profile (dotted line) and buoyancy frequency squared (solid line) from PFP casts at T2, (h) gradient Richardson number from PFP casts at T2.
Table 2.4: Comparison of observations and weakly nonlinear/empirical models for the large amplitude vertical mode two internal waves shown in figures 2.4a,b, 2.5d-f and 2.22c,d (Continuous density and shear model); and 2.6c [Schmidt & Spigel (2000) model and direct observation]. Θ denotes temperature, see text for description of other symbols. The † and ‡ symbols denote values calculated using $T_s = \lambda/c$ and $\lambda = T_s c$, respectively, where the quantities have been obtained as given in this table.

<table>
<thead>
<tr>
<th>Model</th>
<th>Continuous density &amp; shear</th>
<th>Schmidt &amp; Spigel (2000)</th>
<th>Direct observation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_2$ (m)</td>
<td>-</td>
<td>3</td>
<td>-</td>
</tr>
<tr>
<td>$\Theta_1$ (°C)</td>
<td>-</td>
<td>24</td>
<td>-</td>
</tr>
<tr>
<td>$\Theta_3$ (°C)</td>
<td>-</td>
<td>14</td>
<td>-</td>
</tr>
<tr>
<td>$\rho_1$ (kg m$^{-3}$)</td>
<td>-</td>
<td>997.3</td>
<td>-</td>
</tr>
<tr>
<td>$\rho_3$ (kg m$^{-3}$)</td>
<td>-</td>
<td>999.3</td>
<td>-</td>
</tr>
<tr>
<td>$a$ (m)</td>
<td>2.5</td>
<td>2.5</td>
<td>2.5</td>
</tr>
<tr>
<td>$L$ (m)</td>
<td>10±7</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\lambda$ (m)</td>
<td>35±25</td>
<td>13</td>
<td>~65‡</td>
</tr>
<tr>
<td>$c_o$ (m s$^{-1}$)</td>
<td>0.16±0.03</td>
<td>0.085</td>
<td>-</td>
</tr>
<tr>
<td>$c$ (m s$^{-1}$)</td>
<td>0.23±0.001</td>
<td>0.15</td>
<td>0.46±0.001</td>
</tr>
<tr>
<td>$T_s$ (s)</td>
<td>152†</td>
<td>87‡</td>
<td>142±20</td>
</tr>
<tr>
<td>$f_l$ (Hz)</td>
<td>-</td>
<td>-</td>
<td>1.3×10$^{-3}$</td>
</tr>
</tbody>
</table>

2.16b,e), yet coherent waves persist for ~10 h (e.g. figure 2.3).

For $K = 0$, figures 2.16 and 2.17 show the maximum growth rate in wavenumber space to be highest in the direction of the mean flow (figure 2.15). This one-sided advective nature is consistent with what would be expected for an asymmetrical Kelvin-Helmholtz or Holmboe instability (Lawrence et al., 1998). For the V2 wave events (figure 2.16) the frequencies of these unstable modes are between $10^{-3}$ and $10^{-2}$ Hz, which corresponds to the high-frequency band associated with the observed vertically incoherent internal waves. These instabilities have wavelengths between 5–15 m and 15–35 m at T3 and BN50, respectively. The frequency bandwidth and wavelength of these unstable modes are consistent with the observed peak in spectral energy and the phase-coherence observations from the similar wave events observed in Lake Biwa (Tables 2.1 and 2.2). For the V1 wave events (figure 2.17) the maximum growth rate increases exponentially in shape from low values near $10^{-3}$ Hz, to a super-$N$ peak between $10^{-2}$ Hz and $10^{-1}$ Hz followed by a rapid decline thereafter with increasing frequency. The fastest growing instabilities are approxi-
2.7. Linear stability results

Figure 2.15: Plan view of horizontal velocity vectors at: (a) T3 from figure 2.4a,b, (b) BN50 from figure 2.10a, (c) T1 from figure 2.14c and (d) T2 from figure 2.14f. In (a) the black (north) vectors are located through the metalimnion.

mately an order of magnitude in frequency space above the frequency bandwidth of the observed vertically coherent internal waves. At T1 and T2 the most unstable instabilities have wavelengths between 5-10 m and 5-20 m, respectively.

The spectral gap between the observed high-frequency V1 wave events and the unstable modes predicted via linear stability analysis implies a deficiency in the stability model. This discrepancy may not be attributed to the influence of the local $N_c$ cutoff on the observed high-frequency waves. Both the moored thermistor chain observations and the linear stability results are Eulerian with respect to the flow field and are thus subject to Doppler shifting (e.g. Garrett & Munk, 1979).

To examine whether the spurious high-frequency linear instabilities associated with the V1 wave events may be damped by viscosity and diffusion we repeated the linear stability analysis with increased values of $K$. Molecular effects were found to be negligible with no significant change in the wavenumber (or frequency) of the fastest growing modes with $K \leq 10^{-5} \text{ m}^2\text{s}^{-1}$ (not shown). This is in agreement with the work of others for $Re <\sim 300$ (e.g. Nishida & Yoshida, 1987; Smyth et al., 1988). A transition region is identified with $K = 2 \times 10^{-4}$ and $K = 5 \times 10^{-4}$
Figure 2.16: Linear stability results with $K = 0$ (inviscid) for V2 wave events in Lake Kinneret at T3 (figures 2.4a,b and 2.5d-f) and Lake Biwa at BN50 (figures 2.9b and 2.10a-d). Maximum growth rate of unstable modes in horizontal wavenumber space at T3 (a) and BN50 (d). Zonal and meridional wavenumbers are denoted by $k$ and $l$, respectively. Growth rate in frequency ($c_r/\lambda$) space for the most unstable modes at each wavenumber (hollow triangles) at T3 (b) and BN50 (e). Also plotted in (b) and (e) are the 22°C and 17°C, respectively, isotherm displacement spectra as observed from a moored thermistor chain at each location (solid line). The shaded region denotes the region where the growth period $1/\kappa c_i$ is greater than averaging period $\Delta t$ and the dashed vertical line the maximum $N$. The spectra were smoothed in the frequency domain to improve statistical confidence. Maximum growth rate of unstable modes versus wavelength at T3 (c) and BN50 (f). Physical properties of the unstable modes are given in Table 2.2.
2.7. Linear stability results

Figure 2.17: Linear stability results with $K = 0$ (inviscid) for V1 wave events in Lake Kinneret at T1 (figures 2.14a,c-e) and T2 (figures 2.14b,f-h). Maximum growth rate of unstable modes in horizontal wavenumber space at T1 (a) and T2 (d). Zonal and meridional wavenumbers are denoted by $k$ and $l$, respectively. Growth rate in frequency $c_r/\lambda$ space for the most unstable modes at each wavenumber (hollow triangles) at T1 (b) and T2 (e). Also plotted in (b) and (e) are the 27°C isotherm displacement spectra as observed from a moored thermistor chain at each location (solid line). The shaded region denotes the region where the growth period $1/\kappa c_i$ is greater than averaging period $\Delta t$ and the dashed vertical line the maximum $N$. The spectra were smoothed in the frequency domain to improve statistical confidence. Maximum growth rate of unstable modes versus wavelength at T1 (c) and T2 (f). Physical properties of the unstable modes are given in Table 2.2.
Figure 2.18: Same as figure 2.17 except $K = 2 \times 10^{-4}$ m$^2$s$^{-1}$ at T1 and $K = 5 \times 10^{-4}$ m$^2$s$^{-1}$ at T2.
2.7. Linear stability results

Figure 2.19: Same as figure 2.17 except $K = 5 \times 10^{-4} \text{ m}^2 \text{s}^{-1}$ at T1 and $K = 10^{-3} \text{ m}^2 \text{s}^{-1}$ at T2.
Chapter 2. High-frequency internal waves

$m^2 s^{-1}$ at T1 and T2, respectively (figure 2.18). These augmented values of $K$ decrease the region of growth of unstable modes for $\kappa > 2 \text{ rad m}^{-1}$ (figure 2.18a,d). Furthermore, the growth rates of instabilities with frequencies greater than $10^{-2}$ Hz are slightly reduced while those at frequencies less than $10^{-2}$ show an increase in growth rate (figure 2.18b,e). These lower frequency instabilities have wavelengths $\sim 100$ m (figure 2.18c,f). An increase in growth rate with increasing viscosity or with increasing stratification and $Re < 300$ have been shown to be necessary conditions for the primary instabilities to be three-dimensional in character (Smyth & Peltier, 1990). However, at T1 and T2 within the shear layer $Re \sim 10^6$ (figure 2.14), and thus three-dimensional primary instabilities are not expected. As $K$ is further increased to $5 \times 10^{-4}$ and $10^{-3} m^2 s^{-1}$ at T1 and T2, respectively (figure 2.19), the transition becomes complete. The most unstable modes have frequencies near $10^{-3}$ Hz and wavelengths greater than 100 m. However, these modes are physically inconsistent with the phase-coherence observations from Lake Biwa. The dramatic increase in growth rate with increasing $K$ suggests that these solutions may be numerically unstable.

2.7.1 Water column microstructure profiles The horizontal current speed, density and wavefunction profiles for the most unstable modes are shown in figures 2.20 and 2.21. These may be compared to vertical velocity profiles from the PFP casts (figures 2.20b,f and 2.21b,g) which were averaged temporally over $\Delta t$ to minimize fluctuations resulting from irreversible turbulent motions. With $K = 0$ the $\hat{\psi}$ (figures 2.20c,g and 2.21c,h) and observed $w$ profiles show remarkable similarity in the variation of vertical modal structure with depth. Local maximum values of $\hat{\psi}$ are shown to occur within thin layers where $Ri < 1/4$. It is significant to stress that these profiles are independent, that is $\hat{\psi}$ was calculated using density and horizontal velocity profiles only. The most unstable modes with augmented values of $K$ (figure 2.19b,e) exhibit $\hat{\psi}$ profiles (figure 2.21d,i) which are visually inconsistent with the observed variation of vertical modal structure with depth. For these modes the occurrence of local maxima in $\hat{\psi}$ does not appear to be correlated to the local $Ri$.

Although the density profiles are generally similar, the horizontal current speed profiles varied between V1 and V2 events. At T1 and T2 the velocities are greatest in the upper mixed layer and a tendency toward the Kelvin-Helmholtz or Holmboe mode [as determined by comparison of the thickness of the local density/temperature gradient region relative to the local shear gradient region, e.g. Lawrence et al. (1998)] was not evident. At T3 and BN50 horizontal velocities were greatest in jets located within the metalimnion, this vertical mode two structure was also evident in the
2.8 Nonlinear wave model results

We have presented observations of large amplitude waves similar to those produced by nonlinear steepening of basin-scale waves or boundary interaction (e.g. figures 2.7 and 2.9c) and excitation by intrusions and gravity currents (e.g. figure 2.4a), three of the excitation mechanisms described in the introduction.

Equations (2.2), (2.4) and (2.6) were used to calculate the celerity and wavelength of solitary waves which may be supported by the water column profiles shown in figures 2.5d-f and 2.10a-d. Vertical mode one (figure 2.22a,b) and two (figure 2.22c,d) solutions along each of the 32-radial axes were averaged over a range of a. Note the linear dependence of c on a, which accounts for the intrapacket rank ordering of solitary waves by amplitude (e.g. figure 2.9), and the inverse power dependence of λ on a as required by (2.6).

The standard deviation, as indicated by error bars about the mean λ and c in figure 2.22 is a measure of the influence of ambient shear on the wave kinematics at a particular a. This is because with κ = 0 the shear profile input to the linear stability model is the only parameter which is variable along each radial axis. Unlike oceanic observations (e.g. Apel et al., 1985) these results show the wavelength of small amplitude solitary waves and the celerity of large amplitude solitary waves to be strongly modified by in plane vertical shear.

We applied the two and three layer models [(2.6) to (2.9)] to the three large vertical mode one waves of depression ordered according to wave amplitude on the steepened Kelvin wave front between days 247.95 and 248 (figures 2.9c and figure 2.13) and the leading vertical mode two internal solitary waves shown in the wave event of figure 2.6c, respectively. The density of the each layer was estimated from the temperature observations using a standard UNESCO polynomial. Using the
Figure 2.20: Water column profiles from PFP casts and for most unstable modes in figure 2.16. Observed density (broken line) and horizontal velocity (solid line) at T3 (a) and BN50 (e). Observed vertical velocity at T3 (b) and BN50 (f). Vertical velocity eigenfunction $\psi(z)$ for most unstable mode with $K = 0$ (solid line) and depths where $Ri < 1/4$ (shaded) at T3 (c) and BN50 (g). RMS of 0.5 m binned displacements required to monotonize the temperature profile which has been resampled as an average over 0.05 m intervals (ie. $L_T$) at T3 (d) and BN50 (h).
Figure 2.21: Water column profiles from PFP casts and for most unstable modes in figures 2.17 and 2.19. Observed density (broken line) and horizontal velocity (solid line) at T1 (a) and T2 (f). Observed vertical velocity at T1 (b) and T2 (g). Vertical velocity eigenfunction $\hat{\psi}(z)$ for most unstable mode with $K = 0$ (solid line) and depths where $Ri < 1/4$ (shaded) at T1 (c) and T2 (h). Vertical velocity eigenfunction $\hat{\psi}(z)$ for most unstable mode with $K = 5 \times 10^{-4}$ at T1 (d) and $K = 10^{-3}$ at T2 (i) (solid lines) and depths where $Ri < 1/4$ (shaded). RMS of 0.5 m binned displacements required to monotonize the temperature profile which has been resampled as an average over 0.05 m intervals ($L_T$) at T1 (e) and T2 (j).
Figure 2.22: Mean theoretical wavelength ($3.6L$) and mean nonlinear phase velocity of all stable modes over all azimuthal radii where $\kappa$ and $K$ are set to zero. Errorbars denote standard deviation and are an indication of the theoretical influence of mean shear. Vertical mode one wavelength (a) and phase velocity (b) solutions versus soliton amplitude at T3 and BN50. Vertical mode two wavelength (c) and phase velocity (d) solutions versus soliton amplitude at T3 and BN50.
2.9. Discussion

We have presented observations of ubiquitous high-frequency internal wave events. Both large amplitude and sinusoidal waves were observed. The large amplitude waves were found to be reasonably described by nonlinear models. The sinusoidal waves were categorized within two distinct classes: vertically coherent vertical mode one waves (V1 events) which were associated with strong wind shear in the surface layer; and vertically incoherent mode one and mode two internal waves bordering thermocline jets (V2 events). For both classes of sinusoidal waves the local $R_i$ was less than 1/4 and unstable modes were predicted via linear stability analysis - suggesting that the wave packets may be energized through shear instability. Interestingly, the V2 events were well modelled within the frequency domain by the linear stability model; whereas the V1 events were modelled at frequencies an order of magnitude greater than observed. Below we present several theories which address this apparent dichotomy and generalize our results to the context of what is presently known about the energetics and mixing within large stratified lakes.

2.9.1 Coherence and turbulence Microstructure observations in Lake Kinneret have identified two turbulent regimes within the metalimnion (Saggio & Imberger, 2001, their figures 9 and 10): energetic turbulence ($\epsilon \sim 10^{-6} \text{m}^2\text{s}^{-3}$, where $\epsilon$ denotes dissipation of turbulent kinetic energy) and elevated buoyancy flux which was energized by shear production; and less energetic turbulence ($\epsilon \sim 10^{-8} \text{m}^2\text{s}^{-3}$) that was characterized by low strain ratios and very small-scale overturns, indicating that the turbulence was possibly energized by wave-wave interaction. The temporal and spatial locations of these microstructure observations suggests that the V1 wave events are associated with the energetic shear driven turbulence while the V2 wave events correspond to the less energetic small scale turbulence. These observed dissipation rates are used below to calculate dissipation timescales.

A distinguishing feature of the V1 and V2 wave events is that the V1 events are vertically coherent, while the V2 events are not. It follows that the V1 events may result from classical vertically coherent shear instability in an approximate two-layered system (e.g. Lawrence et al., 1998); while the V2 events which occur upon
the internal interfaces of a three-layer mean flow are rendered vertically incoherent as a result of destructive wave-wave interaction between instabilities upon the upper and lower interfaces. The turbulence characteristics from Saggio & Imberger (2001) and the isotherm displacement and velocity measurements presented herein are consistent with this hypothesis.

2.9.2 Growth rate versus decay rate The flow within the metalimnion of Lake Kinneret is predominantly laminar (Saggio & Imberger, 2001). It may thus be argued that the rapidly growing shear instabilities which evolve into turbulent events grow in a laminar environment. Therefore, the application of a ubiquitous ‘eddy viscosity’ to damp the spurious high-frequency modes (ie. $K >> \nu$) may be physically unrealistic. Furthermore, the most unstable modes in figure 2.17 (which have growth rates an order of magnitude greater than those in figure 2.16) may rapidly evolve into gravitationally unstable structures whose degeneration fuels the energetic shear driven turbulence observed by Saggio & Imberger (1998). The influence of dissipation on the selective growth of these unstable modes may be better examined through comparison of the wave decay rate $1/T_D = \epsilon/E_{tot}$ and the wave growth rate $\kappa c_i$ for instabilities of various wavelengths, where $T_D$ is the energy dissipation timescale and $E_{tot}$ is the sum of potential and kinetic energy in the unstable waveform. If the spurious V1 modes predicted through linear stability analysis occur in a frequency bandwidth where the decay rates of the unstable modes are greater than the growth rates, then observed waves are not expected.

During the initial linear stages of growth $E_{tot}$ may be estimated assuming a linear sinusoidal waveform; a waveform demonstrated using both theoretical (Holmboe, 1962; Batchelor, 1967) and direct numerical models (Smyth et al., 1988). From the sinusoidal waveform Holmboe and Kelvin-Helmoltz modes evolve to finite amplitude through harmonic amplification and exponential monotonic growth, respectively (Holmboe, 1962; Batchelor, 1967; Smyth et al., 1988). A linear, two-dimensional sinusoidal wave in a continuously stratified fluid is described in modal form as $w = w_o \cos \kappa x \sin m z \ e^{i\omega t}$, where $w$ is the vertical velocity component (positive upward), $w_o$ is the maximum vertical particle speed, $t$ is time and $\kappa$ and $m$ are the horizontal and vertical wavenumbers, respectively (e.g. Turner, 1973). Following Gill (1982) the potential energy of this vertical mode one wave was approximated through integration of the perturbation density field over one wavelength, where $t$ was set equal to one quarter of the oscillation period such that all energy is in the potential form
\[ E_{\text{tot}} \approx \frac{1}{8} \rho_o w_o^2 \lambda H \left( \frac{\lambda^2}{4H^2} + 1 \right) \]  
\[ (2.10) \]

Here \( H \) is the vertical thickness of the standing wave cell. The dissipation timescale was then calculated as

\[ T_D \approx \frac{E_{\text{tot}}}{\epsilon_v} = \frac{w_o^2}{8 \epsilon} \left( \frac{\lambda^2}{4H^2} + 1 \right) \]  
\[ (2.11) \]

where \( \epsilon_v = \epsilon \rho_o H \lambda \) is the volumetric dissipation rate. This result is consistent with that derived by LeBlond (1966) for linear waves of the form given in equation (2.1).

The decay rate \((1/T_D)\) and growth rate of the fastest growing mode for each wavelength at T1 and T2 with \( K = 0 \) (ie. figure 2.17) are compared in figure 2.23. In equation (2.11) \( H \) and \( w_o \) were obtained from the observed vertical velocity profiles (figure 2.21) while an upper bound of \( \epsilon \sim 10^{-6} \text{ ms}^{-3} \) was estimated using the results of Saggio & Imberger (2001) discussed above. An overlay of the observed spectral density curve from figure 2.17b,e indicates that instabilities at frequencies greater than \( 10^{-2} \) Hz may dissipate their energy at a rate faster than they will grow, hence waves would not be observed at these frequencies. Conversely, waves with frequencies between \( 10^{-3} \) and \( 10^{-2} \) Hz may grow faster than they decay which would result in the growth of instabilities that match the frequency bandwidth of the observed spectral energy peak.

### 2.9.3 Thermocline trapping

An alternative explanation, due simply to internal wave dynamics, has been proposed for the observed rise in spectral energy just below the maximum \( N \) (Mortimer et al., 1968; Garrett & Munk, 1975). For a water column with variable stratification and, in particular, with a thermocline (subsurface region of pronounced maximum \( N \)) internal waves can be trapped within the layer of the water column where \( N \) exceeds the wave frequency (see Groen, 1948; Eckart, 1960; Turner, 1973). This wave trapping has the effect of retaining energy in the region of the frequency domain where \( N_{\text{min}} < f < N_{\text{max}} \) (Eckart, 1960, Chapter 12); hence the observed energy peak just below the maximum \( N \). If the unstable modes predicted via linear stability analysis are oscillatory in nature this trapping will prevent the amplification of the waveforms with frequencies greater than \( N_{\text{max}} \) (Batchelor, 1967, pg. 517). This in turn, will limit the oscillatory waves to the ‘allowed’ \( N_{\text{min}} < f < N_{\text{max}} \) frequency range. Desaubies (1975) has shown that this theoretical model reproduces the sub-\( N \) energy peak observed in oceanic internal wave spectra.
Figure 2.23: Maximum growth rate ($\kappa c_i$) and decay rate ($1/T_D$) of the most unstable mode at each wavelength at T1 (a) and T2 (b) with $K = 0$. Decay rate variables $\epsilon \sim 10^{-6} \text{m}^2 \text{s}^{-3}$, $H = 6 \text{ m}$ and $u_o = 0.04 \text{ m s}^{-1}$ and $0.03 \text{ m s}^{-1}$ at T1 and T2, respectively. Also plotted are the $27^\circ\text{C}$ isotherm displacement spectra as observed from a moored thermistor chain at each location (solid line) and the maximum $N$ (dashed vertical line). The spectra were smoothed in the frequency domain to improve statistical confidence.

### 2.9.4 Eddy coefficients

To determine the appropriateness of the values of $K$ applied in figures 2.18 and 2.19, the $Ri$ based model of Imberger & Yeates (unpubl. data) and the $\epsilon - N$ based models of Osborn (1980) and Barry et al. (2001) were used to calculate estimates of the turbulent diffusivity for mass ($K_\rho$) during the high-frequency wave events (figure 2.24). In these models we applied a constant $\epsilon$ throughout the water column as obtained from the observations by Saggio & Imberger (1998). These observations were within the base of the epilimnion ($10^{-6} \text{ m}^2 \text{s}^{-3}$) and metalimnion ($10^{-8} \text{ m}^2 \text{s}^{-3}$) and therefore the $\epsilon - N$ models are strictly valid through only these regions. The variable nature of $K_\rho$ throughout the water column suggests that our use of a depth-invariant $K$ is physically unrealistic and over-damping may be the cause of the suspected numerical instability when $K$ is large. Regardless, from figure 2.19 the magnitude of $K$ required to suppress the spurious high-frequency instabilities above $10^{-2} \text{ Hz}$ is approximately $5 \times 10^{-4} \text{ m}^2 \text{s}^{-1}$.
and $10^{-3}$ m$^2$ s$^{-1}$ at T1 and T2, respectively. These values of $K$ are of the same order as those predicted at the base of the epilimnion using the three observational models (figure 2.24). At T3 and BN50 $K_\rho$ is below $10^{-5}$ m$^2$ s$^{-1}$ through the metalimnion, the level at which we found no significant influence of $K$ on the unstable modes, and hence our choice of negligible $K$ in our stability simulations for these events is justified.

2.9.5 Energy flux paths Localized shear instability has been observed to be the dominant dissipation mechanism within the Lake Kinneret interior (Saggio & Imberger, 2001). Using simple energy models Imberger (1998) and Saggio & Imberger (1998) demonstrated that this interior metalimnetic dissipation plays a negligible role in the dissipation of basin-scale internal wave energy and is consequently insignificant in the overall energy budget of large stratified lakes. Observations of the spatial distribution of dissipation of turbulent kinetic energy in wind forced lakes (Imberger, 1998; Wüest et al., 2000) show benthic boundary layer (BBL) dissipation to be an order of magnitude more significant than interior dissipation and demonstrates that BBL dissipation is sufficient to damp basin-scale internal waves over the observed time scales of several days.

Conceptual models (e.g. Imberger, 1998), laboratory experiments (e.g. Horn et al., 2000) and field observations (e.g. Gloor et al., 2000; Lemckert et al., 2004) suggest that solitary and higher mode internal waves will propagate to the lake perimeter where they will shoal on sloping boundaries and can lose up to 70% of their energy to the maintenance of the BBL (Michallet & Ivey, 1999; Ivey et al., 2000). In addition to the shoaling of solitary and higher mode internal waves, contributions to dissipation in the BBL also arise from bed shear induced by basin-scale baroclinic currents. Although some models have shown that the bed shear component of BBL dissipation may alone account for this rapid decay of basin-scale internal waves (Fischer et al., 1979; Fricker & Nepf, 2000; Gloor et al., 2000), observational evidence suggests that localized shoaling of internal solitary waves may result in significant energy flux from the basin scale wave field to dissipation and mixing within the BBL (e.g. Michallet & Ivey, 1999). To determine whether the observed high-frequency internal waves may propagate to the lake boundary we evaluated their characteristic horizontal decay lengthscales $L_D$.

For the high-frequency internal waves associated with shear instability, $L_D$ was calculated as the maximum of: $L_D = c_r/\kappa c_i$ - the waves continue to grow until their energy is dissipated through nonlinear billowing; and $L_D = c_i T_D$ - the waves propagate without billowing and are robbed of their energy by the ambient turbu-
Figure 2.24: Profiles of vertical eddy diffusivity ($K_\rho$) calculated using $K_\rho \leq 0.2(\epsilon/N^2)$ (Osborn, 1980), $K_\rho = 0.47(\epsilon/N^2)$ (Barry et al., 2001) and $K_\rho = 3 \times 10^{-5}Ri^{-0.5}$ for $Ri \leq 0.1$ and $K_\rho = 9.3 \times 10^{-12}Ri^{-7}$ for $Ri > 0.1$ (Imberger & Yeates unpubl.). (a) T1, (b) T2, (c) T3 and (d) BN50. $N$ and $Ri$ profiles were determined from PFP casts used as input to the linear stability solver, while $\epsilon$ was estimated from the results of Saggio & Imberger (1998) to be $10^{-6}$ m$^2$s$^{-3}$ at T1 and T2 and $10^{-8}$ m$^{-2}$s$^3$ at T3 and BN50. These estimates of $\epsilon$ are representative of the base of the epilimnion and through the metalimnion and are thus strictly invalid at other depths.
2.9. Discussion

From figure 2.23, $T_D > 1/\kappa c_i$ below the high-frequency spectral energy peak and therefore for the most unstable modes at each wavelength beneath this peak we calculate the maximum $L_D = c_r T_D \approx 637 \pm 425$ m and $428 \pm 219$ m at T1 and T2, respectively. The mean radius of Lake Kinneret is $\sim 10$ km or $20L_D$ indicating that these waves will not reach the lake boundary. This result agrees with the theoretical arguments of LeBlond (1966) that linear internal waves with $\lambda \approx 100$ m will be rapidly attenuated by ambient turbulence.

For the internal solitary waves believed to be excited by nonlinear processes, we applied the simple self-induced shear model by Bogucki & Garrett (1993). The energy within the three large vertical mode one internal solitary waves of depression shown in figure 2.13 was estimated by assuming an interface thickness $5 \text{ m} < h < 10 \text{ m}$ and using the data in Table 2.3. The critical amplitude $a_c \approx 2\sqrt{hh_T} \sim 15 - 22$ m is greater than the observed amplitude, $a \approx 11.4$ m, indicating that the internal solitary waves will not appreciably decay through self-induced interfacial shear as they propagate toward the lake boundary. Energy loss to turbulent eddies which are independent of and small in comparison to the wavelength was neglected (LeBlond, 1966; Bogucki & Garrett, 1993). For each wave the total energy is estimated as $E_{tot} \approx \frac{1}{2}g^' \rho_1 a^2 L \approx 1.6 \times 10^5 \text{ J m}^{-1}$. Assuming the internal solitary wave width scales as the width of the Kelvin wave and using the basin-scale internal wave energy estimates of Imberger (1998) or Saggio & Imberger (1998) results in each internal solitary wave having approximately 1% of the basin-scale vertical mode one Kelvin wave energy. Although this value is quite low, consideration of the number of internal solitary waves in figure 2.9c suggests that in lakes where internal solitary waves are observed (see Horn et al., 2000, and references therein) production and shoaling of these waves may be a significant energy flux path from the basin-scale wave field to the BBL.

The conceptual model suggested by these results may be described as follows. The basin-scale internal wave field, which is energized by surface wind forcing, may be decomposed into the coupled basin-scale components of the baroclinic currents and the baroclinic waves. Degenerative nonlinear processes within the wave domain erode the basin-scale wave energy through the production of solitary waves which, in turn, propagate to the lake boundary where they may shoal and directly energize the BBL. The horizontal currents simultaneously erode the basin-scale wave energy through buoyancy flux and dissipation which result from patchy shear instability within the lake interior and from the baroclinic currents which oscillate across the lake bed.
2.10 Conclusions

We have presented field observations which reveal ubiquitous and sometimes periodic high-frequency internal wave events within two large stratified lakes. Depending on the class of waves, the waves were found to be reasonably described by either linear stability or weakly nonlinear KdV models.

In regions of high shear and low $R_i$, two distinct classes of high-frequency internal waves were observed. Packets of relatively high-frequency and small amplitude vertical mode one waves, vertically coherent in both phase and frequency, were typically observed riding on the crests of basin-scale Kelvin waves and during periods of intense surface wind forcing. These waves manifest a sharp spectral peak just below the local $N$ near $10^{-2}$ Hz. Phase-coherence observations and linear stability analysis suggest that these waves have wavelengths and phase velocities near 30 - 40 m and 15 cm s$^{-1}$, respectively. Irregular lower amplitude internal waves which vary vertically in frequency and phase were also observed in the region of high shear above and below thermocline jets. These waves were accompanied by an abrupt thickening of the metalimnion and manifest a broader spectral peak near $10^{-2}$ Hz. Phase-coherence observations and linear stability analysis suggest that these waves have wavelengths and phase velocities near 10 - 35 m and 5 - 15 cm s$^{-1}$, respectively.

Large amplitude vertical mode one and mode two internal solitary waves were observed near discontinuous density fronts associated with basin-scale wave forcing. These waves excite a broad spectral peak near $10^{-3}$ Hz. Direct observation and results from analytical models suggest that the vertical mode one solitary waves have wavelengths of several hundred meters and phase velocities near 100 cm s$^{-1}$, while the vertical mode two solitary waves have wavelengths of several tens of meters and phase-velocities between 20 and 50 cm s$^{-1}$. The mode one solitary waves were found to each contain approximately 1% of the energy within the basin-scale internal wave field.

The waves associated with shear instability were shown to dissipate their energy within the lake interior, thus possibly accounting for patchy elevated observations of $\epsilon$ within the metalimnion. Conversely, the nonlinear waves were found to be capable of propagating to the lake perimeter where they may shoal, thus releasing their energy directly to the benthic boundary layer via turbulent mixing and dissipation.
CHAPTER 3

The energetics of large-scale internal wave degeneration in lakes

3.1 Abstract

Field observations in lakes, where the effects of the Earth’s rotation can be neglected, suggest that the basin-scale internal wave field may be decomposed into a standing seiche, a progressive nonlinear surge and a dispersive solitary wave packet. In this study we use laboratory experiments to quantify the temporal energy distribution and flux between these three component internal wave modes. The system is subjected to a single forcing event creating available potential energy at time zero (APE). During the first horizontal mode one basin-scale wave period ($T_i$), as much as 10% and 20% of the APE may be found in the solitary waves and surge, respectively. The remainder is contained in the horizontal mode one seiche or lost to viscous dissipation. These findings suggest that linear analytical solutions, which consider only basin-scale wave motions, may significantly underestimate the total energy contained in the internal wave field. Furthermore, linear theories prohibit the development of the progressive nonlinear surge, which serves as a vital link between basin-scale and sub basin-scale motions. The surge receives up to 20% of the APE during a nonlinear steepening phase and, in turn, conveys this energy to the smaller-scale solitary waves as dispersion becomes significant. This temporal energy flux may be quantified in terms of the ratio of the linear and nonlinear terms in the nonlinear nondispersive wave equation. Through estimation of the viscous energy loss, it was established that all inter-modal energy flux occurred before $2T_i$; the modes being independently damped thereafter. The solitary wave energy remained available to propagate to the basin perimeter, where although it is beyond the scope of this study, wave breaking is expected. These results suggest that a periodically forced system with sloping topography, such as a typical lake, may sustain a quasi-steady flux of 20% of APE to the benthic boundary layer at the depth of the metalimnion.

3.2 Introduction

Internal waves were first observed in lakes by Watson (1904) and Wedderburn (1907). They interpreted a temperature oscillation in Loch Ness as a wind-driven unimodal baroclinic standing seiche. Subsequent investigations by Mortimer (1955)
and Thorpe (1971) revealed a progressive component to the Loch’s basin-scale internal wave field. This wave was asymmetrical in character, owing to a steep ‘nonlinear’ wave front and is typically referred to as a progressive internal surge. Remarkably similar observations from other lakes abound (e.g. Hunkins & Fliegel, 1973; Farmer, 1978; Mortimer & Horn, 1982; Boegman et al., 2003). These observations show the internal surge to contain a packet of spatially coherent large-amplitude internal solitary waves which are followed by an oscillatory tail of irregular wavelength. The high-frequency solitary waves are expected to break upon sloping topography at the basin perimeter leading to enhanced dissipation, fluxes and bioproductivity (e.g. Ostrovsky et al., 1996; MacIntyre et al., 1999; Michallet & Ivey, 1999; Kunze et al., 2002). However, the internal solitary waves are dispersive (nonhydrostatic) and sub-grid scale, thus their evolution, propagation and breaking are not reproduced in most coupled hydrodynamic and water quality models (e.g. Hodges et al., 2000; Boegman et al., 2001). At present, these effects may not be parameterized because the distribution and flux of energy between the standing waves, the internal surge and the solitary waves remains unknown (Imberger, 1998; Horn et al., 2001).

In this study we address these issues by using a laboratory model to quantify the temporal energy distribution between the component internal wave modes: the baroclinic standing seiche, the progressive internal surge and the internal solitary waves. We consider systems in which the effects of the Earth’s rotation can be neglected and that are subject to a single forcing event. In section 3.3 we present the relevant theoretical background. The laboratory experiments are described in section 3.4, followed by a presentation of results and a comparison to an analytical model. Finally, in section 3.5 we estimate the loss of energy to the action of viscosity and are thus able to evaluate the modal energy flux. In conclusion, our results are placed within the context of what is presently known about the energetics of stratified lakes.

### 3.3 Theoretical background

During the summer months a stratified lake will typically possess a layered structure consisting of an epilimnion, metalimnion and hypolimnion. If the vertical density gradient is abrupt through the metalimnion, the lake may be approximated as a simple two-layer system of depth $h_1$ and density $\rho_1$ over depth $h_2$ and density $\rho_2$, where $H = h_1 + h_2$ is the total depth and $L$ denotes the basin length (e.g. Heaps & Ramsbottom, 1966; Thorpe, 1971; Farmer, 1978). Internal waves may be initiated within a stratified lake by an external disturbance such as a surface wind stress.
The internal response of the waterbody to a wind stress, as described by Fischer et al. (1979) and Spigel & Imberger (1980), can be gauged by the ratio of the wind disturbance force to the baroclinic restoring force. Thompson & Imberger (1980) introduced this ratio as the Wedderburn number $W$, which may be expressed for our two-layer system (see, Horn et al., 2001) as

$$W^{-1} = \frac{\eta_o}{h_1} \quad (3.1)$$

where $\eta_o$ is the amplitude of the initial disturbance. Note that when $W \approx 1$ the thermocline has upwelled to the surface at the windward shore.

From the initial condition of a tilted thermocline, Horn et al. (2001) applied two-layered theoretical descriptions, verified by laboratory experiments, to identify the mechanisms responsible for the degeneration of the evolving large-scale interfacial gravity wave field. Through comparison of the characteristic timescales of the various degeneration mechanisms, they defined five regimes in which a particular mechanism was expected to dominate (figure 3.1). For small to medium sized lakes subject to weak forcing ($W^{-1} < 0.3$) an internal standing seiche is generated that is eventually damped by viscosity. Moderate forcing ($0.3 < W^{-1} < 1$), results in the production of a progressive internal surge and internal solitary waves.

For weak initial disturbances ($W^{-1} < 0.3$) the standing internal seiche that forms is well described by the linear nondispersive wave equation (e.g. Gill, 1982, pg. 127)

$$\frac{\partial^2 \eta}{\partial t^2} = c_o^2 \frac{\partial^2 \eta}{\partial x^2} \quad (3.2)$$

where $\eta(x,t)$ is the interfacial displacement, $c_o = \sqrt{(g' h_1 h_2)/(h_1 + h_2)}$ the linear long-wave speed and $g' = g (\rho_2 - \rho_1)/\rho_2$ the reduced gravity at the interface. Equation (3.2) admits periodic sinusoidal solutions of the form (e.g. Mortimer, 1974)

$$\eta = a \cos(kx + \omega t) \quad 0 \leq x \leq L \quad (3.3)$$

where $a$ is the wave amplitude, $\omega$ the wave frequency, $k = 2\pi/\lambda$ the wavenumber and $\lambda = 2L$ the fundamental wavelength. The period of an internal seiche $T_i = 2\pi/\omega$ for an enclosed basin is

$$T_i^{(n)} = \frac{2L}{nc_o} \quad (3.4)$$

where $n = 1, 2, 3, \text{etc.}$ denotes the horizontal mode. Hereafter, the fundamental timescale $T_i$, without superscript, will be used to represent the gravest mode where $n = 1$. 

Figure 3.1: Analytical regime diagram characterizing the degeneration of large-scale gravity waves, plotted in terms of the normalized initial forcing scale $\eta_o/h_1$ and the depth ratio $h_1/H$. The laboratory observations are also plotted (⋆, Kelvin-Helmholtz (K-H) billows and bore; ◊, broken undular bore; △, solitary waves; □, steepening; ○, damped linear waves). Reproduced from Horn et al. (2001).

A particular solution to (3.2) may be obtained for a rectangular tank stratified with two superposed incompressible fluids of differing density. The boundary conditions are such that the fluid interface remains perpendicular to the end walls

$$\frac{\partial \eta}{\partial x}(0, t) = \frac{\partial \eta}{\partial x}(L, t) = 0$$

for $t > 0$. The initial conditions consist of a tilted interface with maximum displacement denoted by $\eta_o$ and no initial motion

$$\eta(x, 0) = \frac{2\eta_o}{L} x - \eta_o \quad \frac{\partial \eta}{\partial t}(x, 0) = 0$$
3.3. Theoretical background

for $0 \leq x \leq L$. The available potential energy of the perturbed state at $t = 0$ is
given by integration of the general equation for the available potential energy (e.g.
Cushman-Roisin, 1994, pg. 213)

$$\text{APE} = \frac{g(\rho_2 - \rho_1)}{2} \int_0^L \eta^2(x, t) dx,$$

over the initial condition, leading to

$$\text{APE}_{t=0} = \frac{1}{6} gL(\rho_2 - \rho_1)\eta^2_o.$$

The general solution is

$$\eta(x, t) = \sum_{n=1}^{\infty} k_n \cos \left( \frac{n\pi}{L} x \right) \cos \left( \frac{cn\pi}{L} t \right),$$

where $k_n$ are the coefficients in the Fourier cosine series

$$k_n = \frac{2}{L} \int_0^L \left( \frac{2\eta_o}{L} x - \eta_o \right) \cos \left( \frac{n\pi}{L} x \right) dx \quad n = 1, 3, 5 \ldots$$

$$= \frac{-8\eta_o}{(n\pi)^2}.$$ 

Figure 3.2a-i shows the evolution of the initial condition consisting of component
waves progressing from the boundaries (denoted by arrows). The resultant standing
wave pattern and associated baroclinic flow is periodic for each mode $n$, the period
given by (3.4). At $t = 0$ all wave energy is distributed in the potential form between
the odd modes (figure 3.2j-o). The modal distribution of available potential energy
resulting from the initial condition is given by substituting (3.9) and (3.10) into the
general equation for the available potential energy (3.7), resulting in

$$\text{APE}^{(n)} = 16 \frac{gL(\rho_2 - \rho_1)\eta^2_o}{(n\pi)^4} \quad n = 1, 3, 5 \ldots,$$

which may be summed over $n$ where

$$\sum_{n=0}^{\infty} \frac{16}{[(2n + 1)\pi]^4} = \frac{1}{6},$$

(3.13)
Chapter 3. Internal wave energetics

Figure 3.2: Solution of the wave equation initial value problem resulting from a tilted interface. (a) to (i) evolution of the interface position. Panels are shown over one half internal wave period ($0 < t < T_i/2$) at intervals of $T_i/16$. The flow reverses [ie. (i) to (a)] over $T_i/2 < t < T_i$. (j)-(o) interfacial displacement of horizontal modes $n=1$ through 6, respectively, at $t = 0$ (solid line) and $t = T_i/2$ (broken line). For description of arrows see text.

to give (3.8). This equality between the APE in the initial condition [from (3.6)] and the sum of the APE calculated independently for the horizontal modes [from (3.9) at $t = 0$] demonstrates that the modes given by (3.9) are a complete set.

Comparison of (3.8) and (3.12) reveals that the energy is partitioned between the odd modes with 98.6%, 1.2% and 0.2% of the initial energy in $\text{APE}_{n=1}$, $\text{APE}_{n=3}$ and $\text{APE}_{n=5}$, respectively. For a non-dissipative system, these modal energy distributions represent the sum of kinetic and potential energy and are independent of time.

For moderate initial disturbances ($0.3 < W^{-1} < 1$), nonlinearities become significant and an additional term is required in the governing wave equation. For illustrative simplicity, we consider progressive unidirectional motions described by
the nonlinear and non-dispersive wave equation

\[
\frac{\partial \eta}{\partial t} + c_o \frac{\partial \eta}{\partial x} + \alpha \eta \frac{\partial \eta}{\partial x} = 0
\] (3.14)

where the nonlinear coefficient \( \alpha = \frac{3}{2} c_o (h_1 - h_2)/h_1 h_2 \) (see, for example, Djordjevic \& Redekopp, 1978; Kakutani \& Yamasaki, 1978). Bi-directional propagation and/or end-wall reflection would require a Boussinesq type equation (e.g. New \& Pingree, 2000; Redekopp, 2000; Horn et al., 2002). The dependence of \( \alpha \) on \((h_1 - h_2)\) suggests that the degree of nonlinearity in the system depends on the relative heights of the stratifying layers as well as the local interfacial displacement \( \eta(x,t) \).

Nonlinear waves will steepen under a balance between the unsteady \((\partial \eta/\partial t)\) and nonlinear \(\alpha \eta (\partial \eta/\partial x)\) terms (Long, 1972; Horn et al., 1999). This leads to a steepening timescale \( T_s \) (Horn et al., 2001)

\[
T_s = \frac{L}{\alpha \eta_o}.
\] (3.15)

As a wave steepens, dispersive effects become significant eventually balancing nonlinear steepening (Hammack \& Segur, 1978). This results in the formation of higher-frequency waves of permanent form. These waves often occur in lakes and oceans as localized single entities and are thus called ‘solitary waves’. Internal solitary waves may be modelled to first order in amplitude using the weakly nonlinear Kortweg-de Vries (KdV) equation (e.g. Benney, 1966; Gear \& Grimshaw, 1983; Horn et al., 2002)

\[
\frac{\partial \eta}{\partial t} + c_o \frac{\partial \eta}{\partial x} + \alpha \eta \frac{\partial \eta}{\partial x} + \beta \frac{\partial^3 \eta}{\partial x^3} = 0
\] (3.16)

where the dispersive coefficient \( \beta = \frac{1}{6} c_o h_1 h_2 \). Note that if the interface occurs at mid-depth, \( \alpha \) vanishes thus inhibiting steepening and the subsequent production of solitary waves. A particular solution to (3.16) is the solitary wave (Benney, 1966)

\[
\eta(x - ct) = a \text{sech}^2 \left( \frac{x - ct}{\lambda} \right),
\] (3.17)

where the phase velocity \( c \) and horizontal length scale are given by

\[
c = c_o + \frac{1}{3} a \alpha \quad \lambda^2 = 12 \frac{\beta}{a \alpha^2}.
\] (3.18)

An estimate of the number and amplitude of solitary waves, while beyond the scope of this study, may be obtained from the Schrödinger equation based on inverse scattering theory (e.g. Landau \& Lifshitz, 1959; Whitham, 1974; Apel et al., 1985; Drazin \& Johnson, 1989; Horn et al., 1999).
In general, the internal field that can evolve from the set-up or relaxation of the thermocline is a combination of linear standing seiches, nonlinear progressive surges and dispersive solitary waves. These motions were qualitatively reproduced in the experiments by Thorpe (1971, 1974) and, as described above, the timescales over which these processes occur were derived by Horn et al. (2001). Indeed, the nonlinear and nonhydrostatic nature of the problem does allow derivation of the interfacial displacements resulting from the linear, nonlinear and nonhydrostatic internal wave modes (e.g. Peregrine, 1966; Mortimer, 1974; Horn et al., 1999, 2002); however, the distribution and flux of energy between these modes has remained analytically untractable (Redekopp, 2000). We address this issue in the following sections through the use of laboratory experiments to quantify both the temporal energy distributions and the fluxes.

3.4 Laboratory experiments

3.4.1 Experimental methods Experiments were conducted in a sealed rectangular acrylic tank 6 m long, 0.3 m wide and 0.29 m deep (figure 3.3a). The data were originally collected for the study by Horn et al. (2001). The tank was filled with a two-layer stratification by tilting the tank about its central axis (to a maximum angle of 23° from horizontal) and partially filling the tank with the volume of water required for the upper fresh water layer. From a reservoir of saline water, the lower layer was slowly pumped into the bottom of the tank, thus displacing the buoyant fresh water layer. Once full, the tank was gradually rotated to a horizontal position resulting in a stretching of the isopycnal surface and an increase in the density gradient within the interface. For visualization purposes the interface or one of the two layers was seeded with dye. Prior to commencing an experiment the tank was rotated to the required interfacial displacement angle (figure 3.3b). From this condition, the set-up and subsequent relaxation from a wind stress was simulated through a rapid rotation of the tank to the horizontal position, leaving the interface inclined at the original angle of tilt of the tank (figure 3.3c). Each time the tank was filled with a particular stratification a set of experiments were carried out with increasing angles of tilt. This resulted in a gradual thickening of the density interface over the course of experiments, from about 1 to 2 cm. The resulting vertical displacements of the density interface $\eta(x, t)$ were measured using three ultrasonic wavegauges (Michallet & Barthélemy, 1997) distributed longitudinally along the tank at locations A, B and C. (figure 3.3a). The wavegauges logged data to a personal computer at 50 Hz via a 12-bit analog-to-digital converter. The
3.4. Laboratory experiments

3.4.1 Initial condition with the tank horizontal and the interface inclined

3.4.2 Experimental observations

At the beginning of each experiment the flow was driven by the baroclinic pressure gradient that resulted from the tilted density interface (figure 3.4a). The fluid layers were observed to accelerate rapidly from rest, the lower layer moving toward the downwelled end of the tank and a corresponding return flow in the upper layer. This motion was characteristic of a standing horizontal mode one (H1) seiche (figure 3.4a-f). The progressive surge (figure 3.4c-d) and solitary waves (figure 3.4e-i) were also clearly visible, causing the observed internal wave field to deviate significantly from the linear model (figure 3.2a-i). The progressive surge and solitary wave packet may be qualitatively compared to an undular hydraulic jump (e.g. Henderson, 1966, his figure 6-29a), where the frame of reference is vertically inverted to match the corresponding relative thicknesses of the superposed fluid layers. However, the internal surge is not necessarily the result of supercritical flow conditions, but is usually due to nonlinear steepening of a finite-amplitude wave. The evolution of the progressive surge is further discussed in section 3.5.2.
Chapter 3. Internal wave energetics

Figure 3.4: Video frames showing the standing seiche, evolving progressive surge and solitary waves. The initial condition is shown in panel a. The surge and solitary wave packet are propagating to the left in panels b to f, reflecting off the end-wall in panel g and to the right in panels h to i. For this experiment $h_1/H = 0.3$, $\eta_0/h_1 = 0.9$, $T_i = 110$ s and $T_s = 40$ s.
### 3.4. Laboratory experiments

<table>
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<th>$\theta$ (°)</th>
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Table 3.1: Summary of experimental runs. The experimental variables together with the resolution with which they were determined: the initial angle of tilt $\theta$ (± 0.03°), the interface depth $h_1$ (± 0.2 cm) and the density difference between the upper and lower layers $\Delta \rho \approx 20$ kg m$^{-3}$ (± 2 kg m$^{-3}$).

In figure 3.5a-e interface displacement timeseries observed at wavegauge C (defined in figure 3.3) are compared to the linear analytical solution (3.9). The angle of tilt increases with each successive panel. For small $W^{-1}$ (figure 3.5a), the interface is observed to oscillate about the equilibrium position in the same manner as predicted analytically, although viscous effects cause a continual decrease of amplitude with time. For large $W^{-1}$ (figure 3.5c-e), a secondary wave (denoted by ▽) is observed approximately 30 s out of phase with the H1 seiche (denoted by ▼). This wave is progressive in nature and is described below. As $W^{-1}$ increases, dispersive, viscous and nonlinear processes cause the observed timeseries to deviate strongly from the analytical linear theory solution over shorter timescales. Solitary waves are observed at approximately $T_s$ and viscous decay is evident through the decrease in observed wave amplitude over time. Spectral analysis of the time series at wavegauge C re-
Chapter 3. Internal wave energetics

reveals the most energetic H1 mode as well as a series of odd modes consistent with the analytical model (figure 3.5f).

In figure 3.6a-e, the interface displacement timeseries observed at wavegauge B are compared to the analytical solution. This wavegauge is located near the nodal position for the linear modes and therefore the analytical solution does not exhibit an interfacial displacement. For small $W^{-1}$ (figure 3.6a), the system remains linear, matching the analytical solution. As $W^{-1}$ is increased (figure 3.6b-e), an oscillation with period $T_i/2$ is observed. This oscillation must be a consequence of nonlinear effects and is progressive, reflecting off the vertical end walls and passing the wavegauge twice during $T_i$ (figure 3.4). For large $W^{-1}$ (figure 3.6e) the progressive wave rapidly steepens into an internal surge which degenerates into solitary waves soon thereafter (as $t \to T_s$). Spectra of the interface displacement at wavegauge B (figure 3.6f) confirm the progressive surge signal to have the same $T_i/2$ frequency as the horizontal mode two (H2) mode with harmonics evident at other even modes. In both spectral plots (figures 3.5f and 3.6f) a bifurcation of the individual spectra is evident between approximately $5 \times 10^{-2}$ Hz and $5 \times 10^{-1}$ Hz indicating the frequency bandwidth containing the solitary waves.

The experimental observations reveal the standing seiche, progressive surge and solitary waves occupy discrete bandwidths in frequency space. The discrete nature of the signals thus allows isolation of the component due to each wave group through selective filtering of the interface displacement timeseries in the frequency domain. In particular, the odd linear modes are recorded on wavegauges A and C, the even nonlinear modes on wavegauge B, and solitary waves on all wavegauges at frequencies below $5 \times 10^{-2}$ Hz. The procedure for obtaining the temporal energy distribution in each of these modes is described below.

### 3.4.3 Decomposition of the internal wave field

The basin-scale H1 seiche signal was obtained by low-pass filtering the timeseries from the wavegauges located in non-nodal positions (wavegauges A and C) using a 4th order Butterworth filter with a cutoff frequency located midway between the frequencies of the H1 and H2 modes [i.e. passes frequencies $f < (3/2)T_i$]. This bandwidth neglects the higher linear modes, which contain less than 1.5% of the basin-scale internal wave energy.

The signals associated with the progressive surge and solitary wave modes were observed at all three of the wavegauges. However, these signals were only filtered from the timeseries at wavegauge B, which as a nodal location is not contaminated by the motion of the H1 seiche. The surge and solitary wave signals were isolated using band-pass (surge) and high-pass (solitary waves) 2nd order Butterworth filters
passing frequencies \( (3/2)T_i < f < (1/3)T_i \) and \( f > (1/3)T_i \), respectively. Figure 3.7a-c shows the filtered timeseries for a typical experiment where \( W^{-1} = 0.44 \) and \( h_1/H = 0.2 \). Note the occurrence of solitary waves at \( t \approx T_s \) and the temporal decrease in amplitude of all the observed signals resulting from viscous effects. Comparison of figure 3.7a and figure 3.7b, reveals the progressive surge to be of the same spatial form as the sinusoidal H1 seiche, yet increasing in amplitude as \( t \to T_s \). The progressive nature of the surge results in a doubling of the wave frequency relative to the seiche.

Assuming an equipartition between the kinetic and potential forms of wave energy (e.g. Bogucki & Garrett, 1993), the total energy in each of the three the filtered components was quantified in a particular wave period as twice the potential energy calculated using (3.7). To allow the integral to be evaluated directly from the filtered timeseries, the integrand was transformed from spatial to temporal coordinates using the linear phase speed (e.g. Michallet & Ivey, 1999)

\[
\text{Total energy} = c_0 g \left( \rho_2 - \rho_1 \right) \int_{t_1}^{t_2} \eta^2(t) dt. \tag{3.19}
\]

To apply (3.19) to the filtered standing seiche signal, it was first necessary to uncouple the left and right propagating components making up the standing wave pattern. For these components, the assumption of an equipartition between kinetic and potential wave energy is valid. The total energy was then taken as the sum of the energy in the two components. The uncoupling was accomplished by first calculating the interface displacement \( \eta(x, t) \) for all \( t \) over the tank length \( 0 \leq x \leq L \), through a least squares fit of a cosine function to the filtered interface displacements at wavegauges A, B and C. The maximum standing wave amplitude was then evaluated at the end wall [i.e., \( \eta(x = 0 \text{ or } L, t) \)]. This amplitude was divided by two to give the equivalent amplitudes of the incident and reflected standing wave components. The total energy was then individually calculated for each of these two components using (3.19) and summed. For the progressive surge and solitary waves, \( \eta(t) \) was simply the filtered timeseries. For all three wave groups, the integral in (3.19) was over \( T_i/2 \) as this is the time required for one wave/packet length to propagate through a particular wavegauge. Note that although the filtered timeseries are not orthogonal [i.e. \( \eta = \eta_1 + \eta_2 + \eta_3 \), but \( (\eta_1 + \eta_2 + \eta_3)^2 \neq \eta_1^2 + \eta_2^2 + \eta_3^2 \)], the total wave energy evaluated from the unfiltered (raw) signal remains equal to the sum of the energy evaluated independently for the three filtered modes. This results from the filtered signals being solely a manifestation of the available potential energy in a particular mode,
Figure 3.5: Time series of the observed interface displacement (solid line) and initial value problem (broken line) at wavegauge C: (a) $\eta_o/h_1 = 0.15$; (b) $\eta_o/h_1 = 0.27$; (c) $\eta_o/h_1 = 0.43$; (d) $\eta_o/h_1 = 0.59$; (e) $\eta_o/h_1 = 0.83$. For the time series shown $h_1/H = 0.3$, $T_i = 109$ s and the vertical dotted lines denote $T_s$ from equation (3.15). The ▼ and ▽ symbols denote crests of the observed H1 seiche and progressive surge, respectively. (f) Interface displacement spectra for the observed time series in panels a through e. The bottom line corresponds to the top panel a, etc. and the vertical dotted lines denote the frequencies of the lowest eight basin-scale standing modes calculated from (3.4).

which are subject to the equipartition required by (3.19). Conversely, the unfiltered signal exhibits modal interaction, which may favour the harbouring of energy in the kinetic form and does not necessitate that kinetic and potential energy remain equally partitioned [ie. $C(\eta_1 + \eta_2 + \eta_3)^2 = 2(\eta_1^2 + \eta_2^2 + \eta_3^2)$, for $C \neq 2$].

To verify our methods, a second technique was applied to quantify the energy in the H1 seiche. The cosine function originally fitted to the interface displacements was integrated throughout the domain using (3.7). The total energy was obtained by evaluating the integral at times when the standing wave crests and troughs were at the maximum amplitudes (ie. $t = 0$, $T_i/2$, $T_i/4$, etc.) and all energy was in the potential form. Integrals evaluated at consecutive time intervals (e.g. $t = 0$ and $T_i/2$) were averaged to give a mean energy during that period. This allows for direct
3.4. Laboratory experiments

Comparison with the results from equation (3.19).

To minimize start-up transients, several filter lengths of a reflected copy of the input signal were appended to the beginning of the timeseries. However, transients will still occur over \( t < 1/f_h \), where \( f_h \) is the Nyquist or highest cut-off frequency (shown as a shaded region in Figure 3.7a-c). To compensate for filter start-up with the linear seiche (figure 3.7a), the energy during the first period was taken as the average of the energy in the initial condition, from (3.8), and the H1 energy at \( t = T_i \) using (3.19). Start-up was corrected for in figure 3.7b by replacing the filtered signal with the unfiltered signal within the shaded region. This simple substitution was permissible prior to the evolution of solitary waves at \( t = T_s \). In figure 3.7c a correction for filter start-up was not required as both the filtered and unfiltered signals are near zero prior to the evolution of the solitary waves.

The results of these methods for determining the energy in the various wave groups are shown in figure 3.7d for a typical experimental run. The energy during each integral is normalized by the energy in the initial condition (APE) as determined from the observed displacements at \( t = 0 \) using (3.7). Both energy estimates for the H1 mode compare well, showing an exponential decrease in energy over time from around 65% of APE during the first period to less than 5% after four periods. Surprisingly, 10% of APE is already observed in the progressive surge for
Figure 3.7: Filtered timeseries of the interface displacement for the experiment where $W^{-1} = 0.44$ and $h_1/H = 0.2$. (a) Linear-seiche from low-pass filter $[f < (3/2)T_i]$ of wavegauge A signal, (b) progressive surge from band-pass filter $[(3/2)T_i < f < (1/3)T_i]$ of wavegauge B signal, (c) solitary waves from high-pass filter $[f > (1/3)T_i]$ of signal of wavegauge B signal. Shaded region denotes where transient filter start-up effects were corrected. (d) Temporal evolution of the APE distribution between the wavegroups in panels a to c.
3.4. Laboratory experiments

3.4.4 Experimental results  
The methods to partition the internal wave energy between the linear seiche, progressive surge and internal solitary waves were applied to the data from all experimental runs. The results are presented in figure 3.8, where in each panel the vertical axis is $W^{-1}$ and the horizontal axis $h_1/H$ are analogous to those presented in figure 3.1. In the figure matrix, the three columns represent the three wave groups, the rows show the evolution in time, and the contours in each plot denote the modal energy as a percentage of the observed APE at $t = 0$.

During $0 < t < T_i$ (figure 3.8a,e,i) for large $h_1/H$ and small $\eta_o/h_1$, at least 50-70% of the APE is accounted for by the standing H1 seiche, with less than 10% in the surge or solitary waves. For small $h_1/H$ and large $\eta_o/h_1$, 20-60%, 0-20% and 0-10% of the APE is in the seiche, surge and solitary waves, respectively. The energy content in the H1 seiche thus increases as the system becomes ‘linear’ (ie. $\eta_o/h_1 \to 0$ and $h_1/H \to 0.5$) and decreases uniformly throughout the domain with time (figure 3.8a-d). The partition of energy in the progressive surge (figure 3.8e-h) is greater when the system is ‘nonlinear’ (ie. $\eta_o/h_1 \to 1$ and $h_1/H \to 0$), thus accounting for the reversed energy gradient relative to that of the H1 seiche. Nonlinear steepening initially occurs over $0 < t < T_i$ in the ‘more nonlinear’ region of the domain (figure 3.8e). For these experiments $T_i < T_s < 2T_i$, causing an observed energy flux to the solitary wave mode (panel e to j), rather than energy being conserved in the surge with time (panel e to f). In the ‘less nonlinear’ region, nonlinear and dispersive effects occur over longer timescales $T_i < t < 2T_i$ and $2T_i < t < 3T_i$, respectively, yet the energy flux to the sub basin-scale solitary mode is maintained (the energy flows from panel f to k, rather than f to g). Note that there is little observed energy loss during this dispersive energy transfer from low to high-frequency. These results do not account for the energy loss by the action of viscosity, which eventually damps all motions. Quantification of this loss is required to close the energy budget and so it is theoretically estimated in the following section.
3.5 Discussion

3.5.1 Estimation of viscous damping We have quantified the energy in each of the three component internal wave modes: the basin-scale standing seiche, the progressive surge and the solitary waves. Wave breaking was not observed, which implies that the majority of wave energy is ultimately lost by the action of viscosity. Assuming laminar boundary and interfacial layers, this viscous loss was theoretically estimated for each of the wave modes in isolation.

The fraction of wave energy dissipated from the standing H1 wave during one internal wave period $dE_{H1}/E_{H1}$ was estimated for the laboratory experiments considering both the losses at the solid boundaries and in the interfacial shear layer. Horn et al. (2001) integrated these losses for one wave period over the solid boundaries in each layer (excluding end walls) and the interfacial area (Table 3.2). The first term is the energy loss at the solid boundaries and the second term the loss in the interfacial shear layer. The fraction of progressive surge energy which is lost to viscous damping during one wave period $dE_{NS}/E_{NS}$ was estimated using a theoretical model by Troy (2003). In this formulation the total energy equation is integrated vertically and laterally for progressive interfacial waves in a laterally bounded rectangular channel (Table 3.2). Finally, the rate of energy loss from the solitary waves $dE_{SW}/dt$ was estimated, for a KdV type interfacial solitary wave propagating in a rectangular channel, using the model by Leone et al. (1982) (Table 3.2). Of the three terms on the right hand side of this expression, the first and second represent the energy loss from the lower and upper fluid layers along the rigid boundaries, respectively, and the third term represents the energy loss at the interface. The total energy loss was quantified by summing the loss from each of these three components. This loss was applied to close the energy budget.

In figure 3.9 the axes, columns and rows are analogous to figure 3.8; however, the contours at each time interval denote the sum of both the observed wave energy and the cumulative theoretical viscous loss. Again, the energy is presented as a percentage of the observed APE. Comparison to figure 3.8 reveals that 20%, 10% and 10% of the energy from the seiche, surge and solitary waves, respectively, may be dissipated due to viscosity in the first period alone. Most strikingly, panels c-d, g-h and k-l are nearly identical. Therefore, there is little or no energy flux between modes for $t > 2Ti$; an indication that after the initial nonlinear distribution of energy between the H1 seiche and surge and the subsequent dispersive energy flux to the solitary waves, the internal modes remain uncoupled.
3.5. Discussion

<table>
<thead>
<tr>
<th>Mode</th>
<th>Equation</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>H1 Seiche</td>
<td>$dE_{H1}/E_{H1} \sim \pi \delta_b A_b/V + \nu H T_1/\delta_b h_1 h_2$</td>
<td>Horn et al. (2001)</td>
</tr>
<tr>
<td>Nonlinear surge</td>
<td>$dE_{NS}/E_{NS} \sim (1 - e^{-\sigma_L})$</td>
<td>Troy (2003)</td>
</tr>
<tr>
<td>Solitary waves</td>
<td>$dE_{SW}/dt \sim 4\sqrt{2}\rho_2 (1 - \rho_1/\rho_2) g^{5/4} \nu^{1/2}</td>
<td>\bar{\eta}</td>
</tr>
</tbody>
</table>

Table 3.2: Viscous damping equations. $A_b$ is the total boundary area, $V$ is the tank volume, $\nu$ is the kinematic viscosity and $\delta_b$ is the thickness of the density interface (taken as 1 cm), $\delta_b \approx (\nu T_i/\pi)^{1/2}$ is the thickness of the oscillatory boundary layer (Batchelor, 1967, pgs. 193 and 354). $\sigma$ is the sum of the individual decay rates resulting, in order of dominance, from: sidewall boundary layer dissipation, interfacial dissipation, bottom boundary layer dissipation and internal dissipation (see Troy, 2003, for the decay rate equations). $B$ is the channel width and $\bar{\eta}(t)$ is the maximum solitary wave amplitude (taken as the maximum amplitude in each solitary wave packet) See text for description of other symbols.

3.5.2 The progressive surge An essential feature of the downscale energy flux linking the initial basin-scale motions to the higher-frequency solitary waves is the progressive surge. The amount of energy in the surge is a function of the degree of nonlinearity of the system. This may be expressed as a single nondimensional parameter (rather than $W^{-1}$ and $h_1/H$) taken as the ratio of the linear and nonlinear terms in (3.16)

$$\frac{\alpha \eta_o}{c_o} \equiv \frac{3\eta_o |h_1 - h_2|}{2 h_1 h_2}, \quad (3.20)$$

where we have taken $\eta \sim \eta_o \sim a$. In this formulation, nonlinearities are incorporated due to both increasing $W^{-1}$ and decreasing $h_1/H$. A value of $\alpha a/c_o \sim 0.1$ has been shown to be sufficient for visible nonlinear wave deformation (Holloway & Pelinovsky, 2001, pg. 42). In figure 3.10a, the normalized energy in the surge $E_{NS}/APE$ is plotted versus $\alpha \eta_o/c_o$. Averaged over each wave period the data collapse to a single line, which has been obtained as a least-squares fit (see figure caption for details). For $\alpha \eta_o/c_o > 0.4$, the surge is excited during the first period and the system is sufficiently nonlinear that the maximum energy fraction
Figure 3.8: Temporal evolution of APE distribution between the component internal wave modes. The axes and regime boundaries of each panel are as in figure 3.1.
Figure 3.9: Same as figure 3.8, except viscous energy losses are accumulated with time along each column.
Chapter 3. Internal wave energetics

\[ E_{NS} \approx 0.2. \]  

(3.21)

For \( 0 < \alpha \eta / c_o < 0.4 \) the maximum surge energy is obtained during the second period, because here \( 2T_i > T_s > T_i \). In general, the maximum energy increases in direct proportion to \( \alpha \eta / c_o \) and is given by

\[ E_{NS} \approx 0.5 \frac{\alpha \eta_o}{c_o}. \]  

(3.22)

For \( t > 2T_i \) there is no further energy flux between modes and the surge energy decreases uniformly due to the action of viscosity (in agreement with figure 3.9).

In figure 3.10b, \( E_{NS}/\text{APE} \) contours drawn from the least squares data in figure 3.10a are presented in the \((\alpha \eta_o / c_o) - T_i\) plane. Also plotted on figure 3.10b is the curve of \( T_s \) versus \( \alpha \eta_o / c_o \), showing the \( T_s \) data to collapse to a single line, which is coincident with the ridge of maximum surge energy. The energy in the progressive surge increases to approximately 20\% of APE as \( t \to T_s \), decaying shortly thereafter. These results are consistent with analytical models (3.15) which show the surge to steepen as \( t \to T_s \) and subsequently degenerate into solitary waves. Why the energy in the progressive surge is limited to 20\% of APE remains unknown.

We may investigate the evolution of the progressive surge from the initial condition by conceptualizing the wave field as being composed of two separate parts, both emanating from the positive and negative displacement volumes that initially co-exist but are spatially separated. If we consider displacements of very small amplitude, such as those governed by linearized equations, there is no qualitative distinction between the resulting left and right propagating waves. These components form the standing seiche. However, for larger displacements the negative initial volume will evolve into a packet of solitary waves of depression, while the positive initial volume evolves into a positive dispersive wave, referred to as a rarefaction (Horn et al., 2002). The lack of symmetry between the two finite-amplitude components is a direct consequence of the nonlinearity of the governing equations (e.g. Stoker, 1957, pg. 306) and the progressive nature of the internal surge and solitary wave packet results from the combination of these asymmetrical modes.

3.5.3 Field observations To determine the applicability of our analysis to real lakes, we computed the temporal evolution of the modal energy content for a lake with suitable field data (Baldeggersee, Switzerland). This small deep lake with a rectangular basin shape is subject to pulses of high wind stress along the major axis. Published observations were digitally recovered during a period of seasonal
3.5. Discussion

Figure 3.10: (a) Relationship between the fraction of APE observed as energy in the progressive surge ($E_{NS}$), and the nonlinearity of the system ($\alpha \eta_o/c_o$) over each of the first five $T_i/2$. Squares, diamonds, circles and triangles represent $h_1/H = 0.2, 0.3, 0.4$ and 0.5, respectively. All curves are nonlinear least squares fit. (b) Evolution of $E_{NS}/APE$ in the $\alpha \eta_o/c_o - T_i$ plane. Contour intervals are 2.5%. The dashed line denotes $T_s/T_i$ fitted to pass through values at $h_1/H = 0.2, 0.3, 0.4$ and 0.5.
stratification with three along axis wind events (Lemmin, 1987, his figure 6). These observations occurred at stations positioned in a manner analogous to our distribution of wavegauges, being both near the center of the lake as well as near the boundaries. To allow comparison with our two-layer experiments, the motion of the internal wave field initiated by the wind was isolated as the displacement of the 7° isotherm (located in the center of the metalimnion), and filtered as described in section 3.4.3.

Raw and filtered timeseries (figure 3.11a-b) are qualitatively consistent with the laboratory results (figure 3.8a-b), particularly the relative phase and amplitude of the H1 seiche and surge crests. The H1 seiche energy is initially between 40% and 50% of the APE and as $t \rightarrow T_s$ the energy content in the surge is maximal near 20% of the APE (figure 3.11c). These results are in quantitative agreement with figure 3.8a and e, respectively. Viscous damping appears to be weaker than experimentally observed, with the surge energy persisting at greater than the 10% level for the first six periods, although this may be a consequence of resonant wind forcing. Solitary wave packets were not observed, but such waves would be aliased by the 20 min sampling period. This period is much greater than the theoretical solitary wave period calculated from (3.18), with $a \sim \eta_o$, to be $T_{ISW} = \lambda/c \approx 43/0.14 \approx 300$ s. High-frequency internal solitary wave packets have been observed to occur in other lakes after strong wind forcing events (e.g. Thorpe et al., 1972; Hunkins & Fliegel, 1973; Farmer, 1978; Boegman et al., 2003). These waves are thought to result from nonlinear processes and may each possess $\sim 1\%$ of the energy within the basin-scale internal wave field. Furthermore, the wave packets are capable of propagating to the lake perimeter where they can shoal, thus releasing their energy directly to the benthic boundary layer.

### 3.6 Conclusions

We have extended the work of Thorpe (1971, 1974) and Horn et al. (2001) by quantifying the temporal distribution of energy between the three component internal wave modes. Our model is both qualitatively and quantitatively consistent with published field observations. Depending upon the initial conditions, during $0 < t < T_i$ between 20% and 70% of the APE may be found in the H1 seiche, with less than 20% and 10% in the surge and solitary waves, respectively. The remainder is lost to the action of viscosity. These findings demonstrate that linear analytical models may significantly underestimate the total energy contained in the internal wave field. Furthermore, such linear models can not describe the development of the
Figure 3.11: Raw (---) and filtered (—) timeseries of the $7^\circ$ isotherm displacement from Baldeggersee at (a) station 180 and (b) station 181. The timeseries are filtered as described in section 3.4.3. The lake was subject to three along axis wind events. The first wind event ceases at the origin of the time axis, while the timing of the second and third events is denoted by vertical shading. The initial energy ($\text{APE}_{t=0}$) was evaluated from the maximum isotherm displacement, extrapolated to the lake perimeter, during the forcing events. The ▼ and ▽ symbols denote crests of the H1 seiche and progressive surge, respectively. (c) Temporal evolution of the APE distribution between the H1 seiche and progressive surge. Horizontal shading denotes the expected energy distributions as given by figure 3.8a,e. For this data $h_1 = 10$ m (Lemmin, 1987), $H = 34$ m (Imboden et al., 1983), $\eta_0 \approx 5$ m, $W^{-1} = 0.4$, $h_1/H = 0.3$, $T_i \approx 16$ h and $T_s \approx 30$ h. Note that in panel c $T_s$ (---) is presented from the end of both the first and third wind events.
progressive surge, which serves as a vital link between basin-scale and sub basin-scale motions. The surge receives up to 20% of the APE during the nonlinear steepening phase ($t < T_s$) and, in turn, conveys all of this energy to the smaller scale solitary waves as dispersion becomes significant ($t > T_s$). This temporal energy flux may be quantified in terms of the ratio of the linear and nonlinear terms in the nonlinear nondispersive wave equation. Through estimation of the viscous energy loss, it was established that all modal energy flux occurred while $t < 2T_i$, the modes being independently damped thereafter.

The question remains as to what would occur if the solitary wave packet impinged upon sloping topography at the lake boundary. Results presented in Chapter 4 and the work of Helfrich (1992) and Michallet & Ivey (1999) demonstrate that as much as 90-95% of the energy contained within the solitary wave packet may be lost in a single wave-sloping boundary interaction process, thus enhancing localized mixing and dissipation. Furthermore, if the system were periodically forced it may be imagined that a quasi-steady state is achieved, whereby up to 20% of APE is continually found in the surge and solitary wave modes. This energy would be conveyed to the lake boundary, the particular spatial and temporal energy distributions being governed by the topography and the relative frequencies $1/T_i$, $1/T_s$ and that of the forcing. We leave these outstanding issues to be addressed elsewhere. Finally, it is evident that this degeneration process, which is both nonhydrostatic and sub-grid scale, remains as a challenge to be captured by field-scale hydrodynamic models.
Experiments on shoaling internal waves in closed basins

4.1 Abstract

A laboratory study was performed to quantify the temporal energy flux associated with the degeneration of basin-scale internal waves in closed basins. A down-scale energy transfer was observed from the ‘wind forced’ basin-scale motions to the turbulent motions, where energy was lost due to high-frequency internal wave breaking along sloping topography. The ambient fluid was a two-layer stratification with the interfacial waves being generated at the basin-scale via a single forcing event. Under moderate forcing conditions, steepening of nonlinear basin-scale wave components was found to produce a high-frequency solitary wave packet that contained as much 20% of the available potential energy introduced by the initial condition. The characteristic lengthscale of a particular solitary wave was less than the characteristic slope length, leading to wave breaking along the sloping boundary. The ratio of the steepening timescale required for the evolution of the solitary waves, to the traveltime until the waves shoaled controlled their development and degeneration within the domain. The energy loss along the slope, the mixing efficiency and the breaker type were modelled using appropriate forms of an internal Iribarren number. Defined as the ratio of the boundary slope to the wave slope, this unambiguous parameter allows generalization to the oceanographic context. Comparison with field data supports our results and allows revision of the general picture of the internal wave spectrum in lakes. The portion of the spectrum between motions at the basin and buoyancy scales was found to be composed of progressive waves, both weakly nonlinear waves (sinusoidal profile) with frequencies near $10^{-4}$ Hz and strongly nonlinear waves (sech² profile) near $10^{-3}$ Hz. The results suggest that a periodically forced system may sustain a quasi-steady flux of 20% of the available potential energy to the benthic boundary layer at the depth of the pycnocline.

4.2 Introduction

The turbulent benthic boundary layers located at the perimeter of stratified lakes and oceans serve as vital pathways for diapycnal mixing and transport (Ledwell & Bratkovich, 1995; Ledwell & Hickey, 1995; Goudsmit et al., 1997; Wüst et al., 2000). Within the littoral zone, this mixing drives enhanced local nutrient fluxes

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Chapter 4. Experiments on shoaling internal waves

and bio-productivity (e.g. Sandstrom & Elliott, 1984; Ostrovsky et al., 1996; McIntyre et al., 1999; Kunze et al., 2002). Indirect observations suggest that turbulent benthic boundary layers are energized by internal wave activity. In lakes, the turbulence occurs as a result of (1) bed shear induced by basin-scale currents (Fischer et al., 1979; Fricker & Nepf, 2000; Gloor et al., 2000; Lemckert et al., 2004) and (2) breaking of high-frequency internal waves upon sloping topography at the depth of the metalimnion (Thorpe et al., 1972; MacIntyre et al., 1999; Michallet & Ivey, 1999). While the former process is relatively well understood, the latter remains comparatively unexplored.

In lakes where the effects of the Earth's rotation can be neglected, the internal wave weather may be characterized by the relative magnitudes of the period $T_i$ of the horizontal mode-one internal seiche and the characteristic time-scale of the surface wind stress, $T_w$ (see Stevens & Imberger, 1996, their Table 1 for typical values). When $T_w > T_i/4$, an approach to quasi-equilibrium is achieved, with a general thermocline tilt occurring over the length of the basin (Spigel & Imberger, 1980; Stevens & Imberger, 1996, and figure 4.1b-c). From this initial condition, the resulting internal wave field may be decomposed into a standing seiche, a progressive nonlinear surge and a dispersive high-frequency internal solitary wave (ISW) packet (Boegman et al., 2004). For some lakes, $T_w \ll T_i$ and the progressive surge is believed to originate as a wind-induced wave of depression at the lee-shore (see Farmer, 1978, and figure 4.1d). This wave of depression will progress in the windward direction, steepening as it travels and eventually forming a high-frequency ISW wave packet. In the absence of sloping topography, as much as 20% of the available potential energy (APE) introduced by wind forcing at the basin-scale may be found in the ISW field (Boegman et al., 2004). This downscale energy flux has significant implications for hydrodynamic modelling. The high-frequency waves typically have wavelengths of order 100 m (Boegman et al., 2003); much smaller than the feasible grid spacing of field-scale hydrodynamic models (Hodges et al., 2000). If the high-frequency waves break upon sloping topography, between 5% and 25% (Helfrich, 1992; Michallet & Ivey, 1999) of the incident solitary wave energy (1% to 5% of the APE) may be converted by diapycnal mixing to an irreversible increase in the potential energy of the water column.

The generation and propagation of the progressive surge and ISW packet in natural systems with variable windstress and complex topography are not clearly understood. Many long narrow lakes and reservoirs are characterized by vertical and sloping topography at opposite ends (e.g. at dam walls and river mouths, respec-
4.2. Introduction

Figure 4.1: (a) Schematic diagram of the experimental facility (not to scale). WG denotes ultrasonic wavegauge. See section 4.4 for a detailed description. (b) Initial condition at $t = 0$ with upwelling at the slope. (c) Initial condition at $t = 0$ with downwelling at the slope. (d) Initial condition for the experiments by Michallet & Ivey (1999). Panels b-c and d are representative of $T_w > T_i/4$ and $T_w < T_i/4$, respectively.
tively). The sloping bathymetry is believed to favour breaking of high-frequency waves (Thorpe et al., 1972; Hunkins & Fliegel, 1973) as well as increasing the production of bed shear (Mortimer & Horn, 1982), thus introducing significance to both the direction and magnitude of wind-stress which sets up the initial thermocline displacement. For a complete understanding of the system, this directionality must be coupled with the $T_w$, $T_i$ and the steepening timescale $T_s$ (defined below).

In this study, laboratory experiments and field observations are used to examine the dynamics of the large-scale wave degeneration process in a rectangular stratified basin with sloping topography. Our objectives are to quantify the energy loss from the breaking of the ISW packets as they shoal and to cast our results in terms of parameters which are external to the evolving sub basin-scale flow (e.g. wind speed and direction, boundary slope, quiescent stratification, etc.). This will facilitate engineering application and parameterization into field-scale hydrodynamic models. In section 4.3 we present the relevant theoretical background. The laboratory experiments are described in section 4.4, followed by a presentation of results in section 4.5. To determine the applicability of the analysis, field observations are presented in section 4.6.1. Finally, in sections 4.6.2 and 4.7, the results are summarized and placed within the context of what is presently known about the energetics of stratified lakes.

### 4.3 Theoretical background

During the summer months a stratified lake will typically possess a layered structure consisting of an epilimnion, metalimnion and hypolimnion. If the vertical density gradient is abrupt through the metalimnion, the lake may be approximated as a simple two-layer system of depth $h_1$ and density $\rho_1$ over depth $h_2$ and density $\rho_2$, where $H = h_1 + h_2$ is the total depth and $L$ denotes the basin length (e.g. Heaps & Ramsbottom, 1966; Thorpe, 1971; Farmer, 1978). Internal waves may be initiated within a stratified lake by an external disturbance such as a surface wind stress $\tau$ (e.g. Fischer et al., 1979, pg. 161). This stress advects surface water toward the leeside, thus displacing the internal layer interface through a maximum excursion $\eta_o$ as measured at the ends of the basin. The excursion is dependent upon the strength and duration of the wind event. A steady state tilt of the interface is achieved for $T_w > T_i/4$ allowing $\eta_o$ to be expressed in terms of the shear velocity $u_* = \sqrt{\tau/\rho_o}$ as

$$\eta_o \approx L u_*^2 / g' h_1$$

(4.1)
where \( \rho_o \) is a reference density and \( g' = g(\rho_2 - \rho_1)/\rho_2 \) is the reduced gravity at the interface (see Spigel & Imberger, 1980; Monismith, 1987).

The internal response of the waterbody to a forcing event can be gauged by the ratio of the wind disturbance force to the baroclinic restoring force (Spigel & Imberger, 1980). Thompson & Imberger (1980) quantified this force with the Wedderburn number, which is given for our two-layer system in terms of (4.1) as

\[
W^{-1} = \frac{\eta_o}{h_1}.
\] (4.2)

Weak initial disturbances \((W^{-1} < 0.3)\) will excite a standing seiche that is well described by the linear wave equation (Mortimer, 1974; Fischer et al., 1979; Boegman et al., 2004). The periods of this seiche for a two-layer system are

\[
T_i^{(n)} = \frac{2L}{n c_o},
\] (4.3)

where \( n = 1, 2, 3, \) etc. denotes the horizontal mode (herein the fundamental timescale \( T_i \), without superscript, is used to represent the gravest mode where \( n = 1 \)) and \( c_o = \sqrt{(g' h_1 h_2)/(h_1 + h_2)} \) is the linear long-wave speed.

Moderate forcing \((0.3 < W^{-1} < 1.0)\) results in the development of a nonlinear surge and dispersive solitary wave packet (Thorpe, 1971; Horn et al., 2001; Boegman et al., 2004). The temporal development of the nonlinear surge may be quantified by the nonlinearity parameter defined by Boegman et al. (2004) as

\[
\frac{a \alpha}{c_o} \sim \frac{3\eta_o}{2} \frac{|h_1 - h_2|}{h_1 h_2},
\] (4.4)

where we have taken \( \eta_o \sim a \) and \( \alpha = \frac{3}{2} c_o (h_1 - h_2)/(h_1 h_2) \) (e.g. Djordjevic & Redekopp, 1978; Kakutani & Yamasaki, 1978; Horn et al., 1999). If the interface is at mid-depth, \( \alpha \) vanishes, steepening can not occur and there is no production of solitary waves. As the progressive nonlinear surge steepens, its length scale decreases until nonhydrostatic effects become significant and the wave is subject to dispersion (see Hammack & Segur, 1978). This occurs as \( t \to T_s \), the steepening timescale (Horn et al., 2001)

\[
T_s = \frac{L}{\alpha \eta_o}.
\] (4.5)

Steepening is eventually balanced by dispersion and the surge degenerates into a high-frequency ISW packet. Internal solitary waves may be modelled to first order by the weakly nonlinear Korteweg-de Vries (KdV) equation
Chapter 4. Experiments on shoaling internal waves

\[ \frac{\partial \eta}{\partial t} + c_0 \frac{\partial \eta}{\partial x} + \alpha \frac{\partial^2 \eta}{\partial x^2} + \beta \frac{\partial^3 \eta}{\partial x^3} = 0 \]  \hspace{1cm} (4.6)

where \( \eta(x,t) \) (positive upward) is the interfacial displacement and the dispersive coefficient \( \beta = \frac{1}{6} c_0 h_1 h_2 \). A particular solution to (4.6) is the solitary wave equation (Benney, 1966)

\[ \eta(x - ct) = a \text{sech}^2 \left( \frac{x - ct}{\lambda} \right). \]  \hspace{1cm} (4.7)

where the phase velocity \( c \) and horizontal length scale are given by

\[ c = c_0 + \frac{1}{3} \alpha a \]  \hspace{1cm} (4.8a)

\[ \lambda^2 = \frac{12 \beta}{a \alpha}. \]  \hspace{1cm} (4.8b)

Note the functional dependence between \( \lambda \) and \( a \) for the nonlinear waves.

Breaking of high-frequency ISWs may occur if the waves propagate along the density interface into coastal regions with sloping topography. Off-shore, the interface is typically positioned such that \( h_1 < h_2 \) and \( \alpha < 0 \) resulting in ISWs of depression. As a bounded slope is approached, the shoaling waves will encounter a turning point where \( h_2 = h_1 \) and \( \alpha = 0 \). Beyond the turning point, \( h_1 > h_2 \) and \( \alpha > 0 \) causing the ISW to change polarity and become waves of elevation. Analytical KdV theories (see Miles, 1981, for a review) have been extended to model the evolution and propagation of solitary waves when the topography and background flow are slowly varying (e.g., Lee & Beardsley, 1974; Djordjevic & Redekopp, 1978; Zhou & Grimshaw, 1989). Horn et al. (2000) further extended these theories to a strong space-time varying background, but propagation through the turning point and wave breaking were necessarily avoided. First-order analytical models do not allow transmission of the ISWs through the singular point at \( \alpha = 0 \).

The location of the turning point may be easily obtained for a quiescent two-layer flow with sloping topography in the lower layer. By defining the thickness of the lower layer along the slope in \( x \) as

\[ h_2(x) = H - h_1 - \left( \frac{H}{L_s} \right) x, \]  \hspace{1cm} (4.9)

where \( L_s \) denotes the slope length and \( x = 0 \) at the toe of the slope. The location of the turning point on the slope is given in nondimensional form by letting \( h_2(x) = h_1 \)
4.3. Theoretical background

\[
x \frac{L_s}{L_s} = \frac{h_2 - h_1}{H}.
\]  

(4.10)

As a wave travels along the slope it steepens, \(a\) increases and the streamlines approach vertical. During steepening, the maximum horizontal fluid velocity in the direction of wave propagation \(u_{\text{max}}\) will increase more rapidly than \(c\) and a limiting amplitude may be achieved where the velocities in the wave crest become equal to the phase velocity. Here, this limit is defined as the breaking limit and the location on the slope where this limit is observed is the breaking point.

The type of internal wave breaking that occurs at the breaking point may be inferred by analogy to surface breakers. Of the continuum of surface breaker types that exists, four common classifications are used (see Komar, 1976). Spilling breakers occur on mildly sloping beaches where steep waves gradually peak and cascade down as ‘white water’. Plunging breakers are associated with steeper beaches and waves of intermediate steepness, the shoreward face of the wave becomes vertical, curling over and plunging forward as an intact mass of water. Surging breakers occur on steep beaches where mildly sloping waves peak up as if to plunge, but the base of the wave surges up the beach face. Collapsing breakers, which mark the transition from plunging to surging, begin to curl over and then collapse upon themselves with some of the water mass surging forward up the slope.

For surface waves, Galvin (1968) and Battjes (1974) found that the ratio of the beach slope \(S\) to the wave slope \((a/\lambda)\) was suitable for classifying the breaker type. This ratio is expressed as either off-shore or near-shore forms of the Iribarren number \(\xi\)

\[
\xi_b = \frac{S}{(a_b/\lambda_\infty)^{1/2}}
\]  

(4.11a)

\[
\xi_\infty = \frac{S}{(a_\infty/\lambda_\infty)^{1/2}},
\]  

(4.11b)

where the subscripts \(\infty\) and \(b\) refer to off-shore and near-shore wave properties, respectively (figure 4.2a-b). The breaker height \(a_b\) is measured when the wave face first becomes vertical in the surf zone.

The dynamics of internal wave breaking upon sloping topography have been classified according to the ratio of \(\lambda_\infty\) to the slope length \(L_s\) (e.g. Michallet & Ivey, 1999; Bourgault & Kelley, 2003). This parameter retains no knowledge of the actual boundary slope and consequently may not be used to generalize results.
Figure 4.2: Schematic diagram of wave and slope properties: Off-shore wave amplitude $a_\infty$, off-shore wavelength $\lambda_\infty$, near-shore breaker height $a_b$, boundary slope $S$, slope-length $L_s$ and slope-height $H_1$ and $H_2$.

This is shown in figure 4.2c-d, where in both diagrams $\lambda_\infty/L_s$ is equivalent, yet $H_1/L_s \neq H_2/L_s$ and different breaking dynamics are expected as the internal waves shoal. The utility of the ratio of the beach slope to the internal wave slope has been suggested (e.g. Legg & Adcroft, 2003), but a formal classification based upon internal wave data has yet to be performed. Perhaps, this stems from the difficulty in measuring the parameters $\lambda_b$, $\lambda_\infty$ and $a_\infty$ for internal waves. It is preferable to recast $\xi$ in terms of readily measured variables (e.g. wind speed, quiescent fluid properties). This is accomplished for ISWs with a sech$^2$ profile by noting from (4.4) that $a_\infty/\lambda_\infty \sim \alpha \eta_0/c_o$ resulting in

$$\xi_{\text{sech}} = \frac{S}{(\alpha \eta_0/c_o)^{1/2}}. \quad (4.12)$$

where $\eta_0$ is easily estimated using (4.1). Similarly, for sinusoidal waves $a_\infty/\lambda_\infty \sim f \eta_0/c_o$ giving
\[ \xi_{\text{sin}} = \frac{S}{(f \eta/c_0)^{1/2}}. \]  

(4.13)

In section 4.6.2, the wave frequency \( f \) is discussed for progressive sinusoidal waves.

**4.4 Experimental methods**

The experiments were conducted in a sealed rectangular acrylic tank (600 cm long, 29 cm deep and 30 cm wide) into which a uniform slope of either 1/10 or 3/20 was inserted, extending the entire height of the tank and positioned at one end (figure 4.1). The tank was filled with a two-layer stratification from reservoirs of fresh and saline filtered water (0.45 \( \mu \)m ceramic filter). For visualization purposes the lower layer was seeded with dye (Aeroplane Blue Colour 133, 123). Prior to commencing an experiment the tank was rotated to the required interfacial displacement angle. From this condition, the set-up and subsequent relaxation from a wind stress event was simulated through a rapid rotation of the tank to the horizontal position, leaving the interface inclined at the original angle of tilt of the tank. Depending on the initial direction of rotation prior to commencing an experiment, the resulting inclined interface at \( t = 0 \) could be characterized as either upwelling on the slope (figure 4.1b) or downwelling on the slope (figure 4.1c). The ensuing vertical displacements of the density interface \( \eta(x, t) \) were measured using three ultrasonic wavegauges (Michallet & Barthélémy, 1997) distributed longitudinally along the tank at locations A, B and C (figure 4.1). The wavegauges logged data to a personal computer at 10 Hz via a 16-bit analog-to-digital converter (National Instruments PCI-MIO-16XE-50). The experimental variables considered in this study, together with the resolution with which they were determined are given in Table (4.1).

To visualize the interaction of the internal wave field with the sloping topography, digital images with a resolution of 1 pixel per mm were acquired at a rate of 5 Hz from a progressive scan CCD camera (PULNiX TM-1040) equipped with a manual zoom lens (Navitar ST16160). Individual frames were captured in a LabVIEW environment from a digital framegrabber (National Instruments PCI-1422) and written in real-time to disk. The tank was illuminated with backlight from twenty-eight 12 V 50 W halogen lamps (General Electric Precise MR16; figure 4.1). To remove flicker associated with the AC cycle, a constant DC supply was maintained to the lamps via four 12 V car batteries. The images were corrected for vignetting, variations in illumination intensity, dust and aberrations within the optical system and pixel gain and offset following the methods of Ferrier et al. (1993). This procedure required
Table 4.1: Summary of experimental runs. The experimental variables together with the resolution with which they were determined: the slope $S$, the interface depth $h_1$ ($\pm 0.2$ cm), the maximum interface displacement $\eta_o$ ($\pm 0.2$ cm), the horizontal mode one basin-scale internal wave period ($T_i$), the steepening timescale ($T_s$) and the density difference between the upper and lower layers $\Delta \rho \approx 20$ kg m$^{-3}$ ($\pm 2$ kg m$^{-3}$). The $\eta_o$ values were measured along the vertical end wall, where the + and - symbols denote an initial displacement as measured above and below the quiescent interface depth, respectively (ie. + as in figure 4.1c and - as in figure 4.1d).
flat-field composite images of (1) the room darkened and the lens cap on the camera ('dark' image), as well as (2) the tank filled with dyed saline water from the reservoir ('blue' image) and (3) filtered tap water ('white' image). From the three flat-field image composites, a linear regression was performed on the grayscale response at each pixel location versus the mean pixel response across the CCD array to yield slope and intercept arrays. The experimental images were subsequently corrected at each pixel by subtracting the 'dark' image, subtracting the intercept array and dividing by the slope array (e.g. Cowen et al., 2001). Note that a correction for light attenuation was not required as a result of the backlight geometry. The grayscale pixel response within each image was then calibrated to the fluid density by associating measured densities in the fluid layers with the mean corrected pixel response of the 'blue' and 'white' images and mapping the intermediate values according to the log-linear dye versus concentration relationship as determined from an incremental calibration procedure in a test cell.

4.5 Results

4.5.1 Flow field To illustrate the qualitative features of the flow, let us concentrate on the behaviour of a particular experiment (Run 6, Table 4.1). From the initial condition, as shown in figure 4.3a, the flow was accelerated from rest by the baroclinic pressure gradient generated by the initial tilted density interface. Characteristic of a standing horizontal mode-one (H1) seiche, the lower layer moved toward the downwelled end of the tank with a corresponding return flow in the upper layer. Progressing initially from left to right, an internal surge and ISW packet were also evident (figure 4.3b). These reflected from the vertical end wall and progressed left toward the slope (figure 4.3c). The ISWs subsequently shoaled upon the topographic slope and wave breaking was observed (figure 4.3d,e). Not all wave energy was dissipated at the slope as a long-wave was reflected (figure 4.3e, right side of tank). This was reflected once again from the vertical wall and a secondary breaking event was observed (figure 4.3g). After the secondary breaking event the internal wave field was relatively quiescent (figure 4.3h).

The frequency content of the internal surge and ISW packet, as well as the effects of the directionality associated with the initial forcing condition (ie. upwelling or downwelling on the slope), were examined through a time-frequency analysis carried out using continuous wavelet transforms (Torrence & Compo, 1998). Results from wavegauge B, a nodal location for the H1 seiche, are shown for three representative experiments (figure 4.4), where the forcing amplitude $\eta_o/h_1 \approx 0.7$ and layer thickness
Chapter 4. Experiments on shoaling internal waves

Figure 4.3: Video frames showing the wave field evolving from the initial condition in (a). The surge and ISW packet are propagating to the right in (b), (c) and (f) and to the left in (d), (e) and (g). Wave breaking is shown to occur upon the slope. For this experiment $h_1/H = 0.29$ and $\eta_o/h_1 = 0.90$

$h_1/H \approx 0.2$ were similar. Case 1 is a reference case with no slope (figure 4.4a-b). The majority of the energy was initially found in the progressive surge with a period of $T_i/2$. For $t/T_s \geq 1$, the energy appeared to be transferred directly from the surge to solitary waves with a period near $T_i/16$; there was no evidence of energy cascading through intermediate scales (cf. Horn et al., 2001; Boegman et al., 2004). The ISW packet persisted until $t/T_s > 6$, the wave period gradually increasing from $T_i/16$ to $T_i/8$ as the rank-ordered waves in the packet separate according to (4.8a). The waves were eventually dissipated by viscosity. Case 2 is the initial condition of upwelling on the slope (figure 4.4c-d), as shown in figure 4.1b. The ISW packet initially evolved in the same manner as case 1, for small time. However, over $2 < t/T_s < 3$, the fully developed ISW packet interacted with the sloping topography and rapidly lost its high-frequency energy to wave breaking (see figure 4.3). The reflected long wave, shown progressing to the right at $t/T_s \approx 3$, traveled the tank length and continued steepening. A second ISW packet was observed to develop and was observed to break over $4 < t/T_s < 5$. Case 3 is the initial condition of downwelling on the slope (figure 4.4e,f), as shown in figure 4.1c. In this experiment, the ISW packet initially
interacted with the slope at $t/T_s \approx 1$. At this time the wave packet was in the early stages of formation and only a weak breaking event, with little apparent energy loss from the wave, was observed. The reflected wave packet then continued to steepen as it propagated to the right and reflected off the end wall. Again, a second breaking event was observed over $3 < t/T_s < 4$ with more significant energy loss from the wave. From these results in closed systems, it is clear that the nature of the surge and ISW packet life history is dependent not only upon the bathymetric slope but also on the direction of the forcing relative to the position of the topographic feature.

This concept is illustrated in figure 4.5 where visualization of the laboratory experiments (e.g. figure 4.3) has allowed the wave degeneration process, from the initial conditions shown in figure 4.1b and c, to be conceptually depicted. A long-wave initially propagates from the upwelled fluid volume (panels B and H). This wave steepens as it travels, thus producing high-frequency ISWs. The wave groups reflect from the vertical wall with minimal loss and break as they shoal along the bathymetric slope. The breaking process is expected to be governed by $\xi$.

4.5.2 Internal solitary wave energetics In the proceeding section, a qualitative description of the wave-slope dynamics in a closed basin was presented. To quantify the temporal evolution of the ISW energy and the energy loss from these waves as they interact with the topographic boundary slope, we applied the signal processing methods of Boegman et al. (2004). This technique requires the energy in the various internal modes to be discrete in frequency space (as demonstrated in figure 4.4), thus allowing the individual signals to be isolated through selective filtering of the interface displacement timeseries. By assuming an equipartition between kinetic and potential forms of wave energy, the energy in the surge ($E_{NS}$) and ISW wave packets ($E_{ISW}$) was determined from the filtered timeseries as the integral of the signal over the wave/packet period. Note that net downscale energy flux through wave breaking at the boundary may not be quantified by this method since we can not account for the observed upscale energy flux to reflected long-waves. Results are presented in figure 4.6, showing contours of $E_{NS}/APE$ (panel a) and $E_{ISW}/APE$ (panels b-f). In each panel, the vertical axis ($\alpha \eta_0/c_0$) indicates the relative magnitudes of the linear and nonlinear components of the internal wave field [from (4.4)], while the horizontal axis ($T_i$) reveals how the system evolves in time. The times at which the solitary wave packet shoals upon the sloping beach and $T_s$ are also indicated (see caption).

For case 1, where there is no slope, the surge energy ($E_{NS}/APE$) is shown to increase during an initial nonlinear steepening phase ($0 < t \leq T_s$) and retain up to
Chapter 4. Experiments on shoaling internal waves

Figure 4.4: (a) Timeseries and (b) continuous wavelet transforms showing the temporal evolution of the internal surge and solitary wave packet for the case of no slope (case 1). (c)-(d) same as panels a and b except for the initial condition of upwelling on the 3/20 slope (case 2). (e)-(f) same as panels a and b except for the initial condition of downwelling on the 3/20 slope (case 3). For all experiments $\eta_0/h_1 \approx 0.7$, $h_1/H \approx 0.2$ and $T_s \approx 70$ s. The arrows in panels c and e denote the direction of wave propagation relative to the tank schematics shown in figure 4.1c and d, respectively.
20% of the APE at $t \approx T_s$ (figure 4.6a). When $t > T_s$ the $E_{NS}/\text{APE}$ decreased from this maximum, with a corresponding increase in ISW energy, until eventually all of the surge energy was transferred to the solitary waves with little loss (figure 4.6b). In the absence of sloping topography, the energy in the ISW packet was ultimately lost to viscosity on timescales of order $3T_i$ to $5T_i$.

Figure 4.6c and d show results for case 2, upwelling on boundary slopes of 3/20 and 1/10, respectively. In both panels, the evolution of $E_{ISW}/\text{APE}$ was initially analogous to the case 1 with no slope; however, now the waves shoaled after travel-times of $0.5T_i$, $1.5T_i$ and $2.5T_i$. The first wave/slope interaction occurred at $0.5T_i$, prior to the development of any high-frequency ISWs (ie. $t < T_s$). At this time there was no discernible breaking of the basin-scale surge (with a mild wave slope) on the relatively steep boundary slope. The character of the second wave-slope interaction was dependent upon the magnitude of $\alpha\eta_0/c_o$. For $\alpha\eta_0/c_o > 0.5$ we find $T_s < 1.5T_i$, 

![Figure 4.5: Schematic illustration showing the evolution of the spatial wave profile as observed in the laboratory. (A)-(F) Evolution from the initial condition in figure 4.1c, (G)-(L) Evolution from the initial condition in figure 4.1b. Illustrations are representative of $T_i/2$ time intervals. Arrows denote wave propagation direction.](image-url)
Figure 4.6: General evolution of internal wave energy normalized by the APE introduced at the initial condition: (a) Surge energy (ie. $E_{NS}$/APE) and (b) ISW energy (ie. $E_{ISW}$/APE) for the case of no slope. ISW energy for the case of (c) upwelling along the 3/20 slope and (d) upwelling along the 1/10 slope. ISW energy for the initial condition of (e) downwelling along the 3/20 and (f) downwelling along the 1/10 slope. Contours are presented as a percentage of the APE introduced at $t = 0$ and are compiled using the data in Table 4.1 and Table 3.1. The contour interval is 5%. The ratio $T_s/T_i$ for a particular $\alpha_{\eta_0}/c_o$ is indicated by —. The times at which the solitary wave packet shoals upon the sloping beach are denoted with – –.
which results in nearly all ISW energy (as much as 25% of APE) being lost near the boundary or reflected during the $1.5T_i$ shoaling event. For some experiments where $\alpha \eta_0/c_o \approx 0.5$, secondary breaking events were observed at $2.5T_i$. Experimental visualization revealed that these events transpired when the ISW packet had not fully developed before the primary breaking event (ie. $T_s \approx 1.5T_i$), hence subsequent packet development occurred prior to $2.5T_i$.

In figure 4.6e and f results are shown for case 3, the initial condition of downwelling at boundary slopes of $3/20$ and $1/10$, respectively. Again, the evolution of $E_{ISW}/APE$ is initially similar to the case with no slope but the waves then shoaled around $T_i$ and $2T_i$. By $t \approx T_i$ wave steepening had begun, $E_{ISW}/APE$ energy levels were about 5-10% and this energy was lost from the ISW field at the boundary. As with the case of upwelling at the slope, for $\alpha \eta_0/c_o \approx 0.5$ a secondary breaking event may occur at $2T_i$.

It is clear from figure 4.6 that the forcing direction (upwelling or downwelling) determines the traveltime of the ISW packet until it shoals, as a multiple of $T_i/2$. Similarly, the growth of the ISW packet is governed by $T_s$. The key nondimensional parameter, relative to the traveltime is $T_s/T_i$ (figure 4.6). From (4.3) and (4.5) this parameter is

$$
\frac{T_s}{T_i} = \frac{1}{3} \frac{h_1h_2}{(h_1 - h_2)\eta_o}.
$$

We note that in panels c and d, the maximum $E_{ISW}/APE \approx 20\%$ occurred at $\alpha \eta_0/c_o \approx 0.5$; whereas in panel b, the maximum $E_{ISW}/APE \approx 20\%$ occurred at $\alpha \eta_0/c_o \approx 1$. Experimental visualization suggests that the reduction in wave-field energy as $\alpha \eta_0/c_o \rightarrow 1$ for the experiments with upwelling along the topographic slope may have resulted from enhanced energy transfer to mixing and dissipation as the density interface accelerated along the slope from the initial rest condition (figure 4.7). This mixing did not occur during the experiments with vertical end walls.

4.5.3 Breaker observations Following the success of classifying surface breakers according to the Iribarren number, we found it instructive to test these classifications for internal wave breaking. Measurements of $a_\infty$ and $\lambda_\infty$ were obtained from the wavegauge data at wavegauge B. Images from the CCD camera were used to estimate $a_b$ and classify the breaker type. In figures 4.8, 4.9 and 4.10 internal waves are shown to spill, plunge and collapse, respectively, suggesting that a form of the Iribarren classification may be suitable for internal wave breaking.
Chapter 4. Experiments on shoaling internal waves

Figure 4.7: Images of mixing as the layer interface advances along the slope for $t > 0$. Data from experimental run 6. The horizontal window length is 1 m and the aspect ratio is 1:1. The upper layer density $\rho_1 \approx 1000 \text{ kg m}^{-3}$ and the lower layer density $\rho_2 \approx 1020 \text{ kg m}^{-3}$.

Spilling breakers were observed on the milder slope ($S = 1/10$), when the wave steepness is high ($\alpha \eta_0/c_o > 0.8$). Two examples of spilling breakers are presented in figure 4.8a-f and g-l. In both examples the rear face of each ISW in the packet breaks as a wave of elevation in a spilling manner. Mixing and dissipation appear to be very localized and on occasion (e.g. panel c), baroclinic motions ‘shear off’ the crest of the wave.

Plunging breakers were observed on both slopes for moderately steep waves ($0.4 < \alpha \eta_0/c_o < 0.8$). Plunging occurred primarily during the breaking events at $1.5T_i$ in figure 4.6c and d. The initial condition of upwelling on the slope allowed a long traveltime ($1.5T_i$) and a fully developed wave packet. Recall that in these breaking events up to 20% of the APE is lost from the high frequency ISW field. Two examples of spilling breakers (figure 4.9a-f and g-l), suggest these events are more energetic than the spilling class. In each example, an intact mass of fluid from
4.5. Results

the rear face of each solitary wave of depression was observed to plunge forward (e.g. figure 4.9c and i) becoming gravitationally unstable and forming a core of mixed fluid, which propagates up-slope as a bolus (cf. Helfrich, 1992; Michallet & Ivey, 1999).

Collapsing breakers were also observed to occur on both boundary slopes, when the incident wave slope was milder than that required for plunging. Two examples of collapsing breakers (figure 4.10a-f and g-l) show that the breaking events were less energetic than plunging, and were characterized by a mass of fluid beginning to plunge forward (e.g. figure 4.10d and i) and then collapsing down upon itself (e.g. figure 4.10e and j-k). Compared to plunging breakers, little mixing occurred.

For experiments where \( h_1 = h_2, \alpha = 0 \) and first order KdV type solitary waves are prohibited from forming. Sufficiently strong forcing (\( \eta_0/h_1 > 0.4 \)) may, however, generate supercritical flow conditions and an internal undular bore (see Horn et al., 2001). The large baroclinic shear associated with this forcing was capable of generating Kelvin-Helmholtz instability, both in the tank interior (Horn et al., 2001, their figure 8) and at the boundary (figure 4.11a-f and g-l). Breaking of the undular waves through shear instability was observed to result in strong mixing of the water column (figure 4.11f,l).

4.5.4 Breaker classification and reflection coefficient The breaker visualizations, as presented in figures 4.8 to 4.10, allow classification of the breaker type in terms of \( \xi \) and the measured reflection coefficient \( R = E_r/E_i \), where \( E_r \) and \( E_i \) are the energy in the reflected and incident wave packets, respectively, calculated from the wavegauge B signal. The signal is not filtered to allow for the observed upscale energy transfer from the breaking high-frequency wave packet to a reflected long-wave which scales as the packet length (reflected long-wave energy would be removed by the filter employed in figure 4.6).

In figure 4.12a, b and c we plot \( R \) versus \( \xi_b, \xi_\infty \) and \( \xi_{\text{sech}} \), respectively. For small \( \xi \), the wave slope is steep relative to \( S \) and the wave propagation over the slope approaches that of a wave in a fluid of constant depth, spilling breakers are observed, \( R \to 0 \) and viscosity dominates. The results of Michallet & Ivey (1999) for lone solitary waves (figure 4.12d-e) suggest that for small \( \xi \), minimal wave energy is converted to an increase in the potential energy of the system through diapycnal mixing. Conversely, for waves with very large \( \xi \), the timescale of wave-slope interaction is small, \( R \to 1 \) and the collapsing breakers again induce minimal mixing. Intermediate to these extremes, plunging breakers develop with gravitational instabilities that drive mixing efficiencies peaking near 25%. As the incident waves
Chapter 4. Experiments on shoaling internal waves

Figure 4.8: False colour images of spilling breakers. The horizontal window length is 1 m and the aspect ratio is 1:1. (a)-(f) Run 28; (g)-(l) Run 32. The upper layer density $\rho_1 \approx 1000 \text{ kg m}^{-3}$ and the lower layer density $\rho_2 \approx 1020 \text{ kg m}^{-3}$. 

(a) Time = 101.0 s 
(b) Time = 103.0 s 
(c) Time = 106.0 s 
(d) Time = 107.2 s 
(e) Time = 109.4 s 
(f) Time = 114.0 s 
(g) Time = 81.0 s 
(h) Time = 83.0 s 
(i) Time = 85.2 s 
(j) Time = 87.4 s 
(k) Time = 90.0 s 
(l) Time = 92.0 s
Figure 4.9: False colour images of plunging breakers. The horizontal window length is 1 m and the aspect ratio is 1:1. (a)-(f) Run 6; (g)-(l) Run 12. The upper layer density $\rho_1 \approx 1000$ kg m$^{-3}$ and the lower layer density $\rho_2 \approx 1020$ kg m$^{-3}$. 

(a) Time = 133.4 s  
(b) Time = 134.4 s  
(c) Time = 135.2 s  
(d) Time = 136.4 s  
(e) Time = 138.8 s  
(f) Time = 146.8 s  
(g) Time = 101.8 s  
(h) Time = 103.4 s  
(i) Time = 104.4 s  
(j) Time = 105.2 s  
(k) Time = 106.8 s  
(l) Time = 111.8 s
Figure 4.10: False-colour images of collapsing breakers. The horizontal window length is 1 m and the aspect ratio is 1:1. (a)-(f) Run 14; (g)-(l) Run 11. The upper layer density $\rho_1 \approx 1000 \text{ kg m}^{-3}$ and the lower layer density $\rho_2 \approx 1020 \text{ kg m}^{-3}$. 

(a) Time = 191.4 s  
(b) Time = 195.4 s  
(c) Time = 196.8 s  
(d) Time = 197.6 s  
(e) Time = 199.2 s  
(f) Time = 202.2 s  
(g) Time = 230.8 s  
(h) Time = 232.4 s  
(i) Time = 233.4 s  
(j) Time = 234.8 s  
(k) Time = 235.4 s  
(l) Time = 237.8 s
4.5. Results

(a) Time = 67.6 s
(b) Time = 68.6 s
(c) Time = 69.2 s
(d) Time = 69.8 s
(e) Time = 70.8 s
(f) Time = 73.0 s

(g) Time = 62.4 s
(h) Time = 64.4 s
(i) Time = 67.2 s
(j) Time = 69.2 s
(k) Time = 71.2 s
(l) Time = 72.8 s

Figure 4.11: False colour images of Kelvin-Helmholtz breakers. The horizontal window length is 1 m and the aspect ratio is 1:1. (a)-(f) Run 17; (g)-(l) Run 18. The upper layer density $\rho_1 \approx 1000$ kg m$^{-3}$ and the lower layer density $\rho_2 \approx 1020$ kg m$^{-3}$. 
contain as much as 25% of the APE, it implies that 6% of the APE may be converted by diapycnal mixing to an irreversible increase in potential energy. These results are consistent with those for lone solitary waves by Michallet & Ivey (1999), which have been recast in terms of $\xi$ and are also presented in 4.12b and c. The present results, however, have distinctly lower $R$ values. This may be attributed to the observation that decaying turbulence and wave reflection associated with the leading wave generally interferes with subsequent waves in the packet, increasing fluid straining and therefore dissipation within the breaker. For this same reason, the breaker type classifications strictly apply to only the leading wave in each packet.

4.5.5 The breaking point  Now that breaker classification guidelines have been established and the energy loss at the slope has been quantified, it is useful to estimate the position along the slope where breaking occurs (the breaking point). Consider a thin but finite thickness density profile of the form $\rho(z) \sim \tanh(z)/(\rho_2 - \rho_1)/2$. The horizontal $u$ velocity field is given by (e.g. Lamb, 2002)

$$u \sim c \frac{\partial \zeta}{\partial z} \sim c \frac{\eta \partial \hat{\psi}}{\partial z}$$

(4.15)

where the vertical displacement of the streamline passing through $(x, z)$ relative to its far-field height is $\zeta(x, z, t) \approx \eta(x, t) \hat{\psi}(z)$ and $\hat{\psi}(z)$ is the perturbation stream-function. At the breaking limit, $\eta \sim a$, $\partial \hat{\psi}/\partial z \sim 1/h_1$ and $u_{max} \sim c$ leading to an expression for the wave amplitude

$$a \sim h_1.$$  

(4.16)

The position on the slope where this condition is met, may be found for an initial wave characterized by the off-shore parameters $a_\infty$ and $\lambda_\infty$ using (4.7) through (4.9). Observations presented thus far, show the internal solitary waves of depression to ultimately break as waves of elevation. Before breaking these waves must first pass through the turning point. The analytical difficulties associated with the transmission of a wave through the turning point are avoided by numerically evaluating the equations at discrete locations along $x/L_s$ according to $h_1$ and $h_2(x)$. To pass knowledge of the wave between the discrete locations it is assumed that the volume of fluid displaced by the wave $V$ remains constant (ie. no mixing occurs and the wave-slope interaction is sufficiently rapid that reflection does not occur before wave breaking). Integrating (4.7) in a reference frame moving at the wave speed $c$ yields
4.5. Results

Figure 4.12: Reflection coefficient ($R$), mixing efficiency ($\gamma_{\text{mix}}$) and breaker type classified according to the various forms of the Iribarren number: (a) Breaker type and $R$ versus $\xi_b$, (b) breaker type and $R$ versus $\xi_\infty$, (c) breaker type and $R$ versus $\xi_{\text{sech}}$, (d) breaker type and $\gamma_{\text{mix}}$ versus $\xi_\infty$, and (e) breaker type and $\gamma_{\text{mix}}$ versus $\xi_{\text{sech}}$. The --- lines demarcate the breaker classifications and are inferred in panels d and e from panels b and c, respectively.
Chapter 4. Experiments on shoaling internal waves

\[ V = a \int_0^\lambda \text{sech}^2 \left( \frac{x}{\lambda} \right) dx \tag{4.17} \]

\[ = a\lambda \tanh(1). \tag{4.18} \]

The off-shore wave properties may now be used to evaluate the near-shore properties over \( h_2(x) \) by taking \( V/\tanh(1) = a_\infty \lambda_\infty = a(x)\lambda(x) \). Progressing incrementally along the slope according to (4.9) we discretely evaluate \( c_o(x), \alpha(x) \) and \( \beta(x) \) along \( h_2(x) \) (\( h_1 = \text{constant} \)). Substitution into (4.8b) gives

\[ \lambda(x) = \frac{12\beta}{a a_\infty \lambda_\infty} \tag{4.19} \]

which may be rearranged as

\[ a(x) = \frac{12\beta}{a \lambda^2}. \tag{4.20} \]

Note the transformation of the wavelength-amplitude dependence in (4.8b) to a wavelength-initial condition dependence in (4.19). Equations (4.19) and (4.20) may now be used to determine the location of the breaking limit upon the slope (ie. where \( a = h_1 \)), for a particular wave characterized by \( a_\infty \) and \( \lambda_\infty \).

The evolution of off-shore waves with amplitudes \( a_\infty = -1, -2, ..., -11 \) cm and wavelengths \( \lambda_\infty \) given from \( a_\infty \) by (4.8b) is shown for \( h_1/H = 0.2 \) and \( h_1/H = 0.3 \) in figure 4.13a and b, respectively. From the initial conditions, \( \lambda \) and subsequently \( a \) are calculated at each discrete location along \( x/L_s \). The waves of depression initially steepen along the slope with |a| increasing and \( \lambda \) decreasing. As the waves approach the turning point, \( a \to 0 \) and \( \lambda \to \infty \) as required by (4.19) and (4.20). After the turning point the waves of elevation steepen rapidly and the limiting values of \( \lambda \to 0 \) and \( a \to h_1 \) are approached. Prior to achieving these limits the steepened waves must break.

In figure 4.13c and figure 4.13d we show the first order nonlinear coefficient \( \alpha \) as well as the location of the turning point and the breaking limit from panels a and b. Near the turning point \( \alpha \to 0 \) and the first order nonlinear term in (4.6) becomes small. Second order nonlinearity dominates and the extended KdV model is appropriate; broad waves of low amplitude are expected (e.g. Djordjevic & Redekopp, 1978; Kakutani & Yamasaki, 1978; Grimshaw et al., 1997; Small, 2001). However, broad waves are not observed; possibly because \( \alpha \) increases rapidly after the turning point. The increased role of the second order nonlinear term relative
Figure 4.13: (a)-(b): Theoretical evolution of $\lambda$ (--) and $a$ (--) as waves progress over a uniformly sloping beach. The upper layer thickness $h_1$ (—) remains constant, while the lower layer thickness $h_2$ (···) decreases along the slope for (a) $h_1/H = 0.2$ and (b) $h_1/H = 0.3$. Turning point (---) from equation (4.10), $\alpha$ (--) and breaking limit (—) for (c) $h_1/H = 0.2$ and (f) $h_1/H = 0.3$. The (···) line denotes $\alpha = 0$. Note that the ratio $x/L_s$ renders the results independent of slope.

to the first order nonlinear term, as $\alpha$ changes sign, may thus be to facilitate the transmission of the ISW through the polarity change at the turning point (Grimshaw et al., 1997) and not to significantly modify the wave shape from that described by the first order equations. This conjecture is supported by comparison of the first order breaking limit and the laboratory observations of the breaking point.

The theoretical predictions for the locations of the turning point [from (4.10)] and breaking limit [from (4.16), (4.19) and (4.20)] were found to be consistent with the observed location of the breaking point. For most experimental runs, breaking occurred after the turning point (figure 4.14a) and close to the position of the breaking limit (figure 4.14b), irrespective of the breaker classification. The observed
Figure 4.14: (a) Theoretical turning point versus measured breaking point and (b) theoretical breaking limit versus measured breaking point. (○) spilling breakers, (∆) collapsing breakers, (□) plunging breakers, (⋄) shoaling undular jump. Error bars denote uncertainty in determining the breaking point due to parallax (shows maximum and minimum position).

scatter in the data may be attributed to the motion of the baroclinic H1 seiche, neglected in the theoretical analysis.

4.6 Field observations

4.6.1 Lake Pusiano To assess the applicability of our results we applied our analysis to some recent observations from a lake with suitable field data and topography (Lake Pusiano, Italy; figure 4.15). This small seasonally stratified lake, with near vertical and gradually sloping bathymetry on the north and south shores, respectively, is subject to pulses of high wind stress from the northerly direction (figure 4.16a). The episodic wind events, on days 203 and 205, introduce energy to the lake at the basin-scale. For these events $T_w > T_i/4$ and a general thermocline tilt is expected. Using the observed wind velocity, $W^{-1} = \eta_o/h_i$ was calculated (Table 4.2) from (4.1) and (4.2). Following Stevens & Imberger (1996), the wind stress during each wind event was averaged and integrated over $T_i/4$. The response of the internal wave field to these wind events, as observed at station T, is consistent with other published field observations (e.g. Hunkins & Fliegel, 1973; Farmer, 1978; Mortimer & Horn, 1982; Boegman et al., 2003). Specifically, the generation and subsequent rapid dissipation of high-frequency internal waves was observed (figure 4.16b). To directly compare these observations to the laboratory results, an approximation to a two-layer stratification was made by separating the epilimnion from
the hypolimnion at the 20°C isotherm ($\eta_{20}$). The nonlinear waves were isolated from contamination due to motions at the basin-scale and buoyancy frequency by low-pass filtering and detrending $\eta_{20}$. The total energy was then integrated over the period of a progressive wave packet $T_i/2$

$$E_{total} = c_o g (\rho_2 - \rho_1) \int_{t_o}^{t_o+T_i/2} \eta^2_{20}(t) dt. \quad (4.21)$$

(see, Boegman et al., 2004). The observed temporal evolution of $\eta^2_{20}$ and $E_{total}$ (figure 4.16c and d) are similar to laboratory data from an experiment (run 31) with comparable initial and boundary conditions as shown in figure 4.16e (downwelling on slope, $h_1/H \approx 0.3$ and $0.4 < \eta_o/h_1 < 0.5$). During $0 < t < 0.5T_i$, the field and laboratory data show $E_{total}/APE \approx 20\%$ and $E_{total}/APE \approx 40\%$, respectively. The lower energy levels in the field appear to be attributed to the absence of a progressive nonlinear surge, which in the laboratory carried the high-frequency waves. During $0.5T_i < t < 1.5T_i$, the high-frequency oscillations in both systems contain approximately 15\% of $E_{total}/APE$, and this energy rapidly decreases to approximately 5\% as $t > 1.5T_i$.

Figure 4.5 allows the wave degeneration process in Lake Pusiano to be conceptually depicted. The initial condition, a northerly wind and sloping southern shore, was as shown in panel A. Upon termination of the wind stress, the system evolves schematically as shown in panels B to F. A long-wave initially propagates from the upwelled fluid volume toward the sloping topography (panel B). This wave reflects from the slope and travels back toward the vertical wall (panel C), reflecting once more and steepening as it travels. Upon returning toward the slope, a high-frequency wave packet has formed with $E_{total}/APE \approx 15\%$ (panel D). These waves break upon the slope losing $\sim 10\%$ of the APE to local turbulent dissipation and mixing (panel E).

The observed energy loss at the slope in Pusiano may be compared to the predicted energy loss using the internal Iribarren model put forth in figure 4.12. The nondimensional parameters $\xi_\infty$, $\xi_{sech}$, and $E_r/E_i$ were calculated from the field data on days 203 and 205. In figure 4.12b, the parameter $\xi_\infty$ is shown to be an accurate measure of $E_r/E_i$ in the both the field and in the laboratory. Spilling and plunging breakers are predicted, with the field measurements being grouped amongst the laboratory data collected in this study and that from Michallet & Ivey (1999). Conversely, the parameter $\xi_{sech}$ is shown to over-estimate the energy loss upon the slope in Pusiano relative to the laboratory model in figure 4.12c. Inspection of figure
Figure 4.15: Map showing the position of Lake Pusiano in Italy ($48.80^\circ$N, $9.27^\circ$E) and the lake bathymetry. A thermistor chain (individual sensors every 0.75 m between depths of 0.3 m and 24.3 m) and weather station (2.4 m above the surface) were deployed at station T during July 2003. Data were sampled at 10 s intervals. Isobaths are given in meters.

4.16c and Table 4.2 reveals that the wind forcing is moderate (ie. $W^{-1} \approx 0.4$); consequently the high-frequency waves in Pusiano are sinusoidal in profile and thus will possess a greater wavelength than that given for a sech$^2$ profile by (4.8b). In figure 4.12c, under-estimation of the actual wavelength (and hence over-estimation of the wave slope) will result in a spurious decrease in $\xi_{\text{sech}}$ and corresponding over-estimation of $E_r/E_i$. For these waves $\xi_{\text{sin}}$ is more suitable.

4.6.2 An interpretation of the wave-spectrum A general model for the internal wave spectrum in lakes was proposed by Imberger (1998). This model is analogous to the Garrett-Munk spectrum found in the oceanographic literature. We use the results presented herein and the recent work by Antenucci & Imberger (2001) and Boegman et al. (2003) to update this model for large lakes with a seasonal
4.6. Field observations

Figure 4.16: Observations of wind speed and direction and isotherm displacement from Lake Pusiano at T. (a) Mean hourly wind direction ···, and 10 min average wind speed —, corrected to 10 m above the surface; (b) isotherms at 2°C intervals calculated through linear interpolation of thermistor data at T (10 min average); (c) detail of shaded region in panel b (1 min average). Note change of timebase. The bottom isotherm in panels b and c are 8°C and 10°C, respectively. (d) Normalized instantaneous energy calculated as $c_0 g (\rho_2 - \rho_1) \eta_{20}^2$ from the displacement of the 20°C isotherm ($\eta_{20}$) in panel c and the integral of this quantity shown as a fraction of the APE over $T_i/2$ intervals as shown (see Table 4.2). APE calculated assuming a linear $\eta_{20}$ tilt of maximum excursion given by (4.1). (e) Same as panel d except for the layer interface from experimental run 31 rather than $\eta_{20}$. Letters B through F denote corresponding panels in figure 4.5.
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<th>Source</th>
<th>φ (°)</th>
<th>θ (°)</th>
<th>λ (10^-3 m)</th>
<th>F (Hz)</th>
<th>η (10^-3 m)</th>
<th>ξ (10^-2 m)</th>
<th>R γ mix</th>
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<th>( \nu )</th>
<th>( \alpha )</th>
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4.6. Field observations

Figure 4.17: (a) Spectra of the vertically integrated potential energy signal (see, Antenucci et al., 2000) from Lake Pusiano (figure 4.16b) and lakes Kinneret and Biwa (see Boegman et al., 2003). The data were sampled at 10 s intervals and the spectra have been smoothed in the frequency domain to improve confidence at the 95% level, as shown by the dotted lines. $N_{\text{max}}$ denotes the maximum buoyancy frequency. (b) Observations of wave nonlinearity, profile and frequency from the literature. Sources are as given in Table 4.2. (c) Summary of data presented in panels a and b.
Chapter 4. Experiments on shoaling internal waves

thermocline and a discernible frequency bandwidth between the motions at the basin and buoyancy scales (bandwidth > $10^2 \sim 10^3$ Hz). The three main features of the internal wave spectrum are shown in figure 4.17a. First, the presence of discrete natural and forced basin-scale seiches at frequencies between zero and $\sim 10^{-4}$ Hz. Second, the presence of instabilities, generated by the surface wind forcing and the baroclinic shear of the basin-scale motions in the lake interior, with frequencies $\sim 10^{-2}$ Hz. The third feature is the intermediate portion of the spectrum, which is shown to be dominated by freely propagating nonlinear wave groups capable of breaking at the lake perimeter (depending on $\xi$). Moderate forcing ($0.3 < W^{-1} < 1$), and perhaps topography in rotational systems (Saggio & Imberger, 1998; Wake et al., 2004), is required for excitation of these waves, which are observed to have sinusoidal profiles at $\sim 10^{-4}$ Hz (figure 4.16c) and sech² profiles at $\sim 10^{-3}$ Hz (Saggio & Imberger, 1998; Boegman et al., 2003). It is interesting to note that Lake Kinneret, while strongly forced, does not appear to generate a strong nonlinear internal wave response; yet the spectral peak resulting from shear instability near $\sim 10^{-2}$ Hz is enlarged relative to Biwa and Pusiano (figure 4.17a). Drawing upon examples found in the literature, the behaviour of the spectrum in the nonlinear $10^{-4}$ Hz to $10^{-3}$ Hz bandwidth, may be generalized according to the ratio of the wave height and the depth of the surface layer $a/h_1$ (or equivalently $\eta_o/h_1 = W^{-1}$ since $a \sim \eta_o$). This ratio is commonly used to gauge the nonlinearity of progressive internal waves (e.g. Stanton & Ostrovsky, 1998). A consistent trend is found throughout observations from a variety of natural systems with differing scales. Figure 4.17b shows sinusoidal waves near $10^{-4}$ Hz and sech² waves near $10^{-3}$ Hz when $a/h_1 < 0.4$ and $a/h_1 > 0.4$, respectively. The waves associated with shear instability in Lake Kinneret are mechanistically and observationally inconsistent with this model ($a/h_1 \approx 0.2$ and $\sim 10^{-2}$ Hz). The revised spectral model for stratified lakes is summarized in figure 4.17c.

Combination of the observations in figure 4.17b and the $\xi$ model presented in figure 4.12, suggests that in lakes with moderate to steep boundary slopes $10\% \leq R \leq 50\%$ and $5\% \leq \gamma_{mix} \leq 25\%$, in the Sulu Sea $R \sim 40\%$ and $15\% \leq \gamma_{mix} \leq 20\%$ and in the St. Lawrence estuary $R \sim 6\%$ to $8\%$ (Table 4.2). Note that these measurements may differ by as much as 50% from those calculated using the ratio of the wavelength to the slope length (e.g. Bourgault & Kelley, 2003).
4.7 Conclusions

In stratified waterbodies, internal waves provide the crucial energy transfer between the large-scale motions forced by winds and tides and the small-scale turbulent dissipation and mixing along sloping boundaries. This energy flux occurs through downscale spectral transfer and shoaling of high-frequency internal waves along topography suited for wave breaking. In lakes, moderate wind forcing \((0.3 < \eta_o/h_1 < 1)\) excites sub basin-scale nonlinear wavegroups which propagate throughout the basin. These wavesgroups are also tidally generated in coastal regions. In both systems, when \(0.3 < \eta_o/h_1 \sim a/h_1 < 0.4\) the waves have frequencies near \(10^{-4}\) Hz and a sinusoidal profile, whereas when \(0.4 < \eta_o/h_1 \sim a/h_1 < 1\) the waves have frequencies near \(10^{-3}\) Hz and a \(\text{sech}^2\) profile.

Using laboratory experiments, we have quantified the downscale spectral energy flux for closed basins. High-frequency internal solitary waves evolve from the basin-scale motions as \(t \geq T_s\) and contain up to 20% of the available potential energy introduced by the wind. As \(T_i/2\) is the characteristic wave traveltime over \(L,\) the ratio \(T_i/T_s\) describes the wave evolution and represents a balance between high-frequency wave growth through nonlinear steepening and high-frequency wave degeneration through shoaling at the boundary.

Wave breaking was observed at the boundary in the form of spilling, plunging, collapsing and Kelvin-Helmholtz breakers. A single wave breaking event resulted in 10% to 75% of the incident wave energy being lost to dissipation and mixing, the remaining energy being found in a reflected long wave. Mixing efficiencies ranged between 5% and 25%. Both energy loss and mixing were dependent upon the breaker type. These processes were modelled in terms of an internal Iribarren number \(\xi.\) As the ratio of the wave slope to the boundary slope, the unambiguous definition of \(\xi\) is easily generalized to lakes, oceans and estuaries. For closed basins, knowledge of the wave profile allowed the \(\xi\) to be recast using properties of the quiescent fluid and the forcing dynamics.

Further study is required as the laboratory facility did not permit examination of extremely mild slopes, such as those commonly found in many lakes and coastal oceans. These results will facilitate the parameterization of nonhydrostatic and sub grid-scale wave processes into field-scale hydrodynamic models.
Chapter 4. Experiments on shoaling internal waves
Conclusions

This thesis has quantified the crucial energy transfer, in lakes, between the wind forced wave motions at the basin-scale and the small-scale turbulent mixing and dissipation occurring as a response to wave breaking at the boundary. The transfer takes place through the high-frequency internal wave field. Being nonhydrostatic in character and sub grid-scale, these waves can not be described by practical field-scale hydrodynamic and water-quality models.

Field observations were presented that revealed ubiquitous and sometimes periodic high-frequency internal wave events within two large stratified lakes. Depending on the class of the waves, they were found to be reasonably described by either linear stability theory or weakly nonlinear KdV models. The waves described by linear stability theory were found to excite a spectral energy peak just below the local \( N \), near \( 10^{-2} \) Hz, in regions where \( Ri < \frac{1}{4} \). These waves were observed as packets of (1) vertically coherent small amplitude vertical mode one waves, typically riding on the crests of basin-scale Kelvin waves during periods of intense surface wind forcing, and (2) irregular lower amplitude internal waves which vary vertically in frequency and phase, found in the region of high shear above and below thermocline jets.

The waves associated with shear instability were shown to dissipate their energy near the local generation sites within the lake interior. These waves likely account for published observations of patches of elevated dissipation of turbulent kinetic energy within the metalimnion. However, simple energy models have shown this dissipation to play a negligible role in the overall degeneration of basin-scale internal waves, thus these waves are consequently insignificant in the overall energy budgets of large stratified lakes. Conversely, the nonlinear waves were found to each contain \( \sim 1\% \) of the basin-scale internal wave energy and were capable of propagating to the lake perimeter where they may break, thus releasing their energy directly to
the benthic boundary layer. The results of this investigation prompted laboratory experiments aimed at evaluating the temporal energy flux to the internal solitary wave domain.

For lakes, where the effects of the Earth’s rotation can be neglected, a laboratory model was used to quantify the temporal energy distribution and flux between the three component internal wave modes. Published field studies have shown the internal wave field to be composed of a basin-scale standing seiche, a progressive nonlinear surge and a dispersive internal solitary wave packet. In the laboratory, the system was subjected to a single forcing event, creating available potential energy at time zero. Under moderate forcing conditions, the surge received up to 20% of the available potential energy during a nonlinear steepening phase and, in turn, conveyed this energy to the smaller-scale solitary waves as dispersion became significant. This temporal energy flux may be quantified in terms of the steepening timescale, required for the evolution of the solitary waves, and the ratio of the linear and nonlinear terms in the nonlinear nondispersive wave equation. Through estimation of the viscous energy loss, it was established that all modal energy flux occurred during the first two basin-scale wave periods, the modes being independently damped thereafter.

Further laboratory experiments were undertaken to address the question as to what would occur if the solitary wave packet impinged upon sloping topography at the lake boundary. These experiments were specifically designed to quantify the energy loss from the solitary wave field through the breaking of high-frequency internal waves. For moderate to strong wind forcing ($0.3 < W^{-1} < 1$), sub basin-scale nonlinear wavegroups are excited and propagate throughout the basin. When $0.3 < W^{-1} < 0.4$ these waves have frequencies near $10^{-4}$ Hz and a sinusoidal shape, whereas when $0.4 < W^{-1} < 1$ the waves have frequencies near $10^{-3}$ Hz and a sech$^2$ profile. As described above, the waves evolved from the basin-scale motions at the steepening timescale ($T_s$) and contained up to 20% of the available potential energy introduced by the initial condition. As $T_i/2$ is the characteristic wave traveltime over $L$, the ratio $T_i/T_s$ describes the wave evolution and represents a balance between high-frequency wave growth through nonlinear steepening and high-frequency wave degeneration through shoaling at the boundary.

Wave breaking was observed at the boundary in the form of spilling, plunging, collapsing and Kelvin-Helmholtz breakers. A single wave breaking event resulted in 10% to 75% of the incident wave energy being lost to dissipation and mixing, the remaining energy being found in a reflected long wave. Mixing efficiencies ranged
between 5% and 25%. Both energy loss and mixing were dependent upon the breaker type. These processes were modelled in terms of an internal Iribarren number $\xi$. As the ratio of the wave slope to the boundary slope, the unambiguous definition of $\xi$ is easily generalized to lakes, oceans and estuaries. For closed basins, knowledge of the wave profile allowed the $\xi$ to be recast in a manner more suitable for engineering applications using properties of the quiescent fluid and the forcing dynamics.

The conceptual model suggested by these results may be described as follows. The basin-scale internal wave field, which is energized by surface wind forcing, may be decomposed into the coupled basin-scale components of the baroclinic currents and the baroclinic waves. Degenerative nonlinear processes within the wave domain erode the basin-scale wave energy through the production of progressive waves, which propagate to the lake boundary. The baroclinic currents simultaneously erode the basin-scale wave energy through buoyancy flux and dissipation which result from patchy shear instability within the lake interior and from the currents which oscillate across the lake bed. The freely propagating nonlinear waves fill the portion of the energy spectrum between the motions at the basin-scale and the near buoyancy frequency instabilities. These waves carry considerable energy to the lake perimeter, which is lost to dissipation and mixing as the waves shoal. The ratio of the boundary slope to the wave slope controls the dynamics of the breaking process.

Both in the field and in the laboratory, the wave groups were observed to rapidly dissipate, thus suggesting that a periodically forced system with sloping topography will sustain a quasi-steady flux of 20% of the forcing energy to the benthic boundary layer at the depth of the metalimnion. Further study is required as the laboratory facility did not permit examination of extremely mild slopes, such as those commonly found in lakes and coastal oceans. These results will facilitate the parameterization of nonhydrostatic and sub grid-scale wave processes into field-scale hydrodynamic models.
APPENDIX A

Derivation of the equations governing two-dimensional internal waves in stratified flows

A.1 Abstract

Three fundamental solutions for two-dimensional internal waves in continuously stratified flows are derived. These solutions are well known and describe waves ranging from the large-scale to the buoyancy-scale. The solutions are: the Taylor-Goldstein equation, the linear long wave equation and the Korteweg-de Vries equation.

A.2 Fundamental equations

- Conservation of density equation

\[ \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + w \frac{\partial \rho}{\partial z} = 0 \]  (A.1)

- X-momentum equation (no rotation)

\[ \rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho w \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} \]  (A.2)

- Z-momentum equation

\[ \rho \frac{\partial w}{\partial t} + \rho u \frac{\partial w}{\partial x} + \rho w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} - \rho g \]  (A.3)

We begin by applying a standard perturbation decomposition to the state variables

\[ u = \bar{u}(z) + \hat{u}(x, z, t) \]
\[ w = \hat{w}(x, z, t) \]
\[ \rho = \bar{\rho}(z) + \hat{\rho}(x, z, t) \]
\[ p = \bar{p}(z) + \hat{p}(x, z, t) \]

and cross-differentiating to derive a single momentum equation. Substituting the perturbations into (A.1) gives

\[ \frac{\partial \hat{\rho}}{\partial t} + \bar{u} \frac{\partial \hat{\rho}}{\partial x} + \hat{u} \frac{\partial \hat{\rho}}{\partial x} + \hat{w} \frac{\partial \hat{\rho}}{\partial z} + \bar{w} \frac{\partial \hat{\rho}}{\partial z} = 0 \]  (A.4)
where subscripts denote differentiation. Substituting the perturbations into (A.2)

\[
(\ddot{\rho} + \dot{\rho})[\ddot{u} + (\bar{u} + \tilde{u})\dot{u} + \dot{\bar{w}}(\bar{u} + \tilde{u})_z] = -\ddot{\bar{p}}_x
\]  

(A.5)

where \(\ddot{u}, \dot{u}_x\) and \(\ddot{\bar{p}}_x\) equal zero by definition. Substituting perturbations into (A.3)

\[
(\ddot{\rho} + \dot{\rho})[\ddot{w} + (\bar{u} + \tilde{u})\dot{\bar{w}} + \dot{\bar{w}}(\bar{u} + \tilde{u})_z] = -(\ddot{\bar{p}} + \dot{\bar{p}})_z - (\ddot{\bar{p}} + \dot{\bar{p}})g
\]  

(A.6)

We cross-differentiate (A.5) and (A.6) to remove the pressure terms. Taking \(\partial(A.5)/\partial z\) gives

\[
\{(\ddot{\rho} + \dot{\rho})[\ddot{u} + (\bar{u} + \tilde{u})\ddot{u}_x + \dot{\bar{w}}(\bar{u} + \tilde{u})_z]\}_z = -\ddot{\bar{p}}_{xz}
\]  

(A.7)

and \(\partial(A.6)/\partial x\) gives

\[
\{(\ddot{\rho} + \dot{\rho})[\ddot{w} + (\bar{u} + \tilde{u})\ddot{u}_x + \dot{\bar{w}}(\bar{u} + \tilde{u})_z]\}_x = -\ddot{\bar{p}}_{zx} - g\ddot{\bar{p}}_x
\]  

(A.8)

We now subtract (A.8) from (A.7). Since we are only interested in keeping 1st order nonlinear terms we immediately drop all terms involving the multiplication of three perturbation quantities. These terms will be very small. Equations (A.7) and (A.8) now reduce to a single momentum equation

\[
\begin{align*}
(\ddot{\rho} + \dot{\rho})(\dddot{u} + \ddot{\bar{u}}\dddot{u}_x + \dddot{\bar{u}}\dddot{u}_z)_z + [\ddot{\bar{p}}(\dddot{u}\dddot{u}_x + \dddot{\bar{w}}\dddot{u}_z)]_z \\
- [(\ddot{\rho} + \dot{\rho})(\dddot{u}_x + \dddot{\bar{w}}\dddot{u}_z)]_x - [\ddot{\bar{p}}(\dddot{u}\dddot{w}_x + \dddot{\bar{w}}\dddot{w}_z)]_x \\
- g\ddot{\bar{p}}_x = 0
\end{align*}
\]  

(A.9)

Equations (A.4) and (A.9) may not be simply solved. These equations are fully non-hydrostatic, non-Boussinesq, weakly nonlinear and subject to a continuous mean vertical shear. Through the relaxation of one or more of these conditions three fundamental solutions to these equations are derived below. Typically (A.4) and (A.9) are solved for particular known \(\bar{u}\) and \(\bar{\rho}\) profiles by using the streamfunction \((\psi)\) to eliminate \(\tilde{u}\) and \(\tilde{w}\) which results in a set of two equations and two unknowns, \(\psi\) and \(\ddot{\bar{\rho}}\). The fundamental solutions are

- **The Taylor-Goldstein equation**: Linear wave modes of finite wavelength which are unstable to mean shear
  - keep only the linear terms, apply the Boussinesq approximation and look for exponentially growing solutions of finite wavelength
• **Linear long wave equation**: Linear and stable long wave modes  
  – keep only the linear terms and look for solutions as the wavenumber approaches zero

• **Korteweg-de Vries equation**: Weakly nonlinear and stable long wave modes  
  – keep the linear and 1st order nonlinear terms and look for solutions as the wavenumber approaches zero

### A.3 The Taylor-Goldstein equation

We look for linear wave modes of finite wavelength which are unstable to mean shear. Keeping the linear terms of (A.4) and (A.9) gives

\[ \tilde{\rho}_t + \tilde{u}\tilde{\rho}_x + \tilde{w}\tilde{\rho}_z = 0 \]  \hspace{1cm} (A.10)

and

\[ [\tilde{\rho}(\tilde{u}_t + \tilde{w}_z\tilde{w})]_z - [\tilde{\rho}(\tilde{w}_t + \tilde{u}\tilde{w}_x)]_x - \frac{g}{\rho_o} \tilde{\rho}_x = 0 \]  \hspace{1cm} (A.11)

We simplify the equations further by applying the Boussinesq approximation. To do so we partition \( \tilde{\rho}(z) = \rho_o + R(z) \) (cf., Fischer et al., 1979, pg. 155) and divide (A.11) by \( \rho_o \).

\[ [(\rho_o + R)(\tilde{u}_t + \tilde{u}\tilde{u}_x + \tilde{u}_z\tilde{w})]_z - [(\rho_o + R)(\tilde{w}_t + \tilde{w}_z\tilde{w})]_x - \frac{g}{\rho_o} \tilde{\rho}_x = 0 \]  \hspace{1cm} (A.12)

Boussinesq assumed that \( R/\rho_o << 1 \) when not multiplied by \( g \) which results in

\[ (\tilde{u}_t + \tilde{u}\tilde{u}_x + \tilde{u}_z\tilde{w})_z - (\tilde{w}_t + \tilde{u}\tilde{w}_x)_x - \frac{g}{\rho_o} \tilde{\rho}_x = 0. \]  \hspace{1cm} (A.13)

Defining the buoyancy frequency as \( N^2 = -[g/(\rho_o + R)](\rho_o + R)_z \) which reduces to \( N^2 = -(g/\rho_o)\tilde{\rho}_z \) in the limit of the Boussinesq approximation allows (A.10) to be simplified to

\[ \tilde{\rho}_t + \tilde{u}\tilde{\rho}_x - \frac{\rho_o}{g} N^2 \tilde{w} = 0 \]  \hspace{1cm} (A.14)

We now reduce the number of unknowns by defining the perturbation stream function \( \psi \) as
\[ \tilde{u} = \psi_z \quad \tilde{w} = -\psi_x \]  
(A.15)

Equations (A.13) and (A.14) reduce to

\[ (\psi_{zz} + \psi_{xx})_t + \tilde{u}\psi_{zzz} + \tilde{w}\psi_{xxx} - \psi_x \tilde{u}_{zz} = \frac{g}{\rho_o} \tilde{p}_x \]  
(A.16)

and

\[ \tilde{\rho}_t + \tilde{u}\tilde{\rho}_x + \frac{\rho_o}{g} N^2 \psi_x = 0 \]  
(A.17)

We look for exponentially growing solutions of the form

\[ \psi(x, z, t) = \hat{\psi}(z) \exp[ik(x - ct)] \]  
(A.18)

\[ \tilde{\rho}(x, z, t) = \hat{\rho}(z) \exp[ik(x - ct)] \]  
(A.19)

where \( k \) is the horizontal wavenumber and \( c \) is the linear wave speed. Subbing (A.18) and (A.19) into (A.17) gives

\[ -\hat{\rho}ikc + \tilde{u}\hat{\rho}ik + \frac{\rho_o}{g} N^2 \hat{\psi}k = 0 \]  
(A.20)

\[ \hat{\rho}(\tilde{u} - c) + \frac{\rho_o}{g} N^2 \hat{\psi} = 0 \]  
(A.21)

\[ \hat{\rho} = -\frac{\rho_o N^2}{g(\tilde{u} - c)} \hat{\psi} \]  
(A.22)

Now subbing (A.18), (A.19) and (A.22) and into (A.16)

\[ -c\hat{\psi}_{zz} - i^2 k^2 c\hat{\psi} + \tilde{u}\hat{\psi}_{zz} + i^2 k^2 \tilde{w}\hat{\psi} - \tilde{u}_{zz}\hat{\psi} = -\frac{N^2}{(\tilde{u} - c)} \hat{\psi} \]  
(A.23)

taking \( i^2 = -1 \) and rearranging results in the Taylor-Goldstein equation

\[ \hat{\psi}_{zz} + \left[ \frac{N^2}{(\tilde{u} - c)^2} - \frac{\tilde{u}_{zz}}{(\tilde{u} - c)} - k^2 \right] \hat{\psi} = 0 \]  
(A.24)

This eigenvalue equation may be solved numerically using a matrix (e.g., Winters & Riley, 1992; Hogg et al., 2001) or shooting (e.g., Sun et al., 1998) method. In both techniques \( N(z), \tilde{u}(z) \) and \( k \) are specified variables which are used to solve for the eigenfunction \( \hat{\psi}(z) \) and the complex eigenvalue \( c = c_r + c_i \). The growthrate of a
A.4. The linear wave equation

The equations are now simplified to describe the profiles of linear and stable long waves. If rotation is included these equations may be used to model Kelvin and Poincaré waves. We are interested in long waves only so take the wavenumber $k = 0$. Neglecting vertical mean shear from (A.10) and (A.11) and applying the perturbation stream function gives

$$
\tilde{\rho}_t - \psi_x \tilde{\rho}_z = 0 \quad (A.25)
$$

$$
(\tilde{\rho} \psi_{zt})_z + (\tilde{\rho} \psi_{zt})_x - g \tilde{\rho}_x = 0 \quad (A.26)
$$

We now look for stable solutions of the form

$$
\psi(x, z, t) = A(x, t) \hat{\psi}(z) + H.O.T. \quad (A.27)
$$

$$
\tilde{\rho}(x, z, t) = A(x, t) \hat{\rho}(z) + H.O.T. \quad (A.28)
$$

Note that $k$ has been set to zero and that since we are presently looking for a wave profile equation we have introduced the wave profile function $A(x, t)$. For linear waves $A$ is typically a sinusoidal function. Substitution of (A.27) and (A.28) into (A.25) and (A.26) gives

$$
A_t \hat{\rho} - A_x \hat{\psi} \hat{\rho}_z = 0 \quad (A.29)
$$

$$
A_t (\tilde{\rho} \hat{\psi}_z)_z + (\tilde{\rho} \psi_{zt})_x - A_x \hat{\rho} g = 0 \quad (A.30)
$$

The second term of (A.30) can be neglected since it is shown in (A.23) to be proportional to $k^2$. We will later show this explicitly. With this simplification these equations are separable. If the separation constant is chosen to be $-c$ then

$$
\frac{A_t}{A_x} = \frac{\tilde{\rho}_x \hat{\psi}}{\hat{\rho}} = -c \quad (A.31)
$$
\[
\frac{A_t}{A_x} = \frac{\hat{\rho} g}{(\hat{\rho} \psi_z)_z} = -c \tag{A.32}
\]

The resulting linear wave equation and eigenfunction equation are

\[
A_t + c A_x = 0 \tag{A.33}
\]

and

\[
(\hat{\rho} \psi_z)_z + \frac{\hat{\rho} N^2}{c^2} \psi = 0 \tag{A.34}
\]

where \(c\) physically represents the linear wave speed. Note that \(N\) is in the non-Boussinesq form \(N^2 = -(g/\hat{\rho})\hat{\rho}_z\). However, if we apply the Boussinesq approximation to (A.34) we see that it is equivalent to (A.24) with \(k\) and \(\bar{u}\) equal to zero. This result is consistent with the specific assumptions used in the derivation of (A.24) and (A.34) from (A.4) and (A.9).

An analytical solution of \(c\) and \(\hat{\psi}\) for linear long waves may be derived by assuming \(N\) is constant and \(\bar{u}, k = 0\). Equation (A.34) may be reduced to a second order linear homogeneous differential equation by applying the Boussinesq approximation. This requires that \(|H \hat{\rho}_z| \ll \hat{\rho}\) where \(H\) is the water depth.

\[
\hat{\psi}_{zz} + \frac{N^2}{c^2} \hat{\psi} = 0 \tag{A.35}
\]

The equation may be solved using the characteristic equation (e.g. Boyce & DiPrima, 1992, pg. 139).

\[
\hat{\psi}(z) = C_1 \cos \left( \frac{N}{c} z \right) + C_2 \sin \left( \frac{N}{c} z \right) \tag{A.36}
\]

Our boundary conditions are \(\hat{\psi}(0) = 0, \hat{\psi}(H) = 0, \hat{\psi}_{zz}(0) = 0\) and \(\hat{\psi}_{zz}(H) = 0\). For a nontrivial solution these give

\[
C_1 = 0, \quad C_2 \neq 0 \tag{A.37}
\]

so

\[
\hat{\psi}(z) = C_2 \sin \left( \frac{N}{c} z \right) \tag{A.38}
\]

and
A.5. The Korteweg-de Vries equation

\[ \sin \left( \frac{NH}{c} \right) = 0 \quad (A.39) \]

Therefore,

\[ \hat{\psi}_n(z) = \sin \left( \frac{n\pi}{H} z \right) \quad n = 0, 1, 2, \ldots \quad (A.40) \]

\[ c_n = \pm \frac{NH}{n\pi} \quad n = 0, 1, 2, \ldots \quad (A.41) \]

These results show that \( C_2 \) is an arbitrary constant which renders the magnitude of the wavefunction indeterminate. \( \hat{\psi}_n(z) \) has thus been defined as a normalized variable such that \(-1 < \hat{\psi}_n < 1\) for \( n = 0, 1, 2, \ldots \).

We may plot (A.40) from \( z = 0 \) to \( z = H \) to determine the structure of the wavefunction for each vertical mode \( n \) (figure A.1). Note that \( n = 0 \) is the surface gravity wave mode which is described by

\[ c_0 \approx \sqrt{gH} \quad \hat{\psi}_0 \approx \frac{z + H}{H} \quad (A.42) \]

and clearly not equations (A.40) and (A.41).

A.5 The Korteweg-de Vries equation

The Korteweg-de Vries (KdV) equation describes weakly nonlinear and stable long wave modes (e.g. solitary waves). In the derivation we retain the linear and 1st order nonlinear terms and search for solutions as \( k \to 0 \). To evaluate which terms scale with \( k \) and may thus be neglected along with the higher order nonlinear terms the KdV equation is typically transformed through appropriate scaling into nondimensional form. Here we follow this practice and the scales are used in our nondimensionalizations are

- Horizontal length scale of waves, \( L \)
- Vertical length scale of water column, \( H \)
- Characteristic wave amplitude, \( \eta \)
- Small parameter, \( \epsilon = \eta/H \)
- Small parameter, \( \mu^2 = H^2/L^2 \sim H^2k^2 \)
Appendix A. Two-dimensional internal wave equations

Figure A.1: Wavefunctions of vertical internal modes one through three

- Horizontal velocity scale, \( U = \sqrt{gH} \)
- Vertical velocity scale, \( W = UH/L \)
- Reference density, \( \rho_o \)

We define the following nondimensional variables

\[
\begin{align*}
    x' &= x/L, \quad z' = z/H \\
    \tilde{u}' &= \tilde{u}/\epsilon U, \quad \tilde{w}' = \tilde{w}/\epsilon W \\
    \tilde{\rho}' &= \tilde{\rho}/\epsilon \rho_o, \quad \tilde{\rho}' = \tilde{\rho}/\rho_o \\
    \tilde{p}' &= \tilde{p}/\epsilon \rho_o gH, \quad \tilde{p}' = \tilde{p}/\rho_o gH \\
    t' &= tU/L 
\end{align*}
\]

We now substitute the above variables into (A.4) and (A.9). Furthermore, to streamline the cumbersome derivation we neglect the mean flow which may act to modify the waveshape. This maintains the essence of the problem, however, by retaining the nonlinear and dispersive terms. Equations (A.4) and (A.9) reduce to
A.5. The Korteweg-de Vries equation

\[ \epsilon[\tilde{\rho}_t + \tilde{w}'\tilde{\rho}_z + \epsilon(\tilde{u}'\tilde{\rho}_x + \tilde{w}'\tilde{\rho}_z)] = 0 \] (A.43)

and

\[
\begin{align*}
[(\tilde{\rho}' + \epsilon\tilde{\rho}')\epsilon\tilde{u}_t]'_z &+ [\epsilon^2\tilde{\rho}'(\tilde{u}'\tilde{u}_x' + \tilde{w}'\tilde{u}_z')]_z \\
-[(\tilde{\rho}' + \epsilon\tilde{\rho}')\epsilon\mu^2\tilde{w}_t]'_x &- [\epsilon^2\mu^2\tilde{\rho}'(\tilde{u}'\tilde{w}_x' + \tilde{w}'\tilde{w}_z')]_x \\
-\epsilon\tilde{\rho}'_x &= 0
\end{align*} \] (A.44)

We can drop all terms with an order higher than \( \epsilon^2 \) (i.e., multiplication of three perturbation quantities) or \( \mu^2 \) (i.e., scale with \( H^2/L^2 \) since \( k \to 0 \)) and (A.44) reduces to

\[
\begin{align*}
[(\tilde{\rho}' + \epsilon\tilde{\rho}')\epsilon\tilde{u}_t]'_z &+ [\epsilon^2\tilde{\rho}'(\tilde{u}'\tilde{u}_x' + \tilde{w}'\tilde{u}_z')]_z \\
-\epsilon\mu^2\tilde{\rho}'_x &= 0
\end{align*} \] (A.45)

We now apply the perturbation streamfunction to the nondimensional equations (A.43) and (A.45). For ease of notation the primes used to denote nondimensional quantities are dropped and both equations are divided by \( \epsilon \). Note that \( \epsilon \) is now reassigned from indicating a single perturbation quantity to represent the multiplication of two perturbation quantities or weakly nonlinear terms.

\[ \tilde{\rho}_t - \psi_x\tilde{\rho}_z + \epsilon(\psi_z\tilde{\rho}_x - \psi_x\tilde{\rho}_z) = 0 \] (A.46)

and

\[ (\tilde{\rho}\psi_{zt})_z + \epsilon[\tilde{\rho}\psi_{zt} + \tilde{\rho}(\psi_z\psi_{xz} - \psi_x\psi_{zz})]_z + \mu^2\tilde{\rho}\psi_{xzt} - \tilde{\rho}_x = 0 \] (A.47)

Again we look for stable solutions of the form

\[ \psi(x, z, t) = A\hat{\psi}^{(0,0)} + \epsilon A^2\hat{\psi}^{(1,0)} + \mu^2 A_{x\hat{\rho}}\hat{\psi}^{(0,1)} + H.O.T. \] (A.48)

\[ \tilde{\rho}(x, z, t) = A\hat{\rho}^{(0,0)} + \epsilon A^2\hat{\rho}^{(1,0)} + \mu^2 A_{x\hat{\rho}}\hat{\rho}^{(0,1)} + H.O.T. \] (A.49)

where \( A = A(x, t), \hat{\psi} = \hat{\psi}(z), \hat{\rho} = \hat{\rho}(z) \) and the superscripts \( (i, j) \) refer to the order of \( \epsilon \) and \( \mu^2 \), respectively. Henceforth for notational ease we drop the superscripts and must therefore exercise caution in manipulations involving \( \hat{\psi} \) and \( \hat{\rho} \).
Substitution of the expansions into the momentum equation (A.47), elimination the higher order terms and expansion of the derivatives gives

\[
\bar{\rho}_z (A_t \hat{\psi}_z + 2 \epsilon AA_t \hat{\psi}_z + \mu^2 A_{zzt} \hat{\psi}_z) + \bar{\rho}(A_t \hat{\psi}_{zz} + 2 \epsilon AA_t \hat{\psi}_{zz} + \mu^2 A_{zzt} \hat{\psi}_{zz})
- (A_x \hat{\rho} + 2 \epsilon AA_x \hat{\rho} + \mu^2 A_{xxx} \hat{\rho}) + \epsilon AA_t (\hat{\rho} \hat{\psi}_z)_z + \epsilon AA_x [\hat{\rho} (\hat{\psi}_z)_z]^2
- \epsilon AA_x (\hat{\rho} \hat{\psi}_{zz})_z + \mu^2 A_{zzt} \hat{\rho} \hat{\psi} = 0 \quad (A.50)
\]

Similarly, substituting the expansions into the density equation (A.46) eliminating the higher order terms and expanding the derivatives gives

\[
A_t \hat{\rho} + 2 \epsilon AA_t \hat{\rho} + \mu^2 A_{zzt} \hat{\rho} - A_x \bar{\rho}_z \hat{\psi} - 2 \epsilon AA_x \bar{\rho}_z \hat{\psi}
- \mu^2 A_{xxx} \bar{\rho}_z \hat{\psi} + \epsilon AA_x \bar{\rho}_z \hat{\psi} = 0 \quad (A.51)
\]

We now rearrange terms and substitute the nondimensionalized linear wave solution

\[
A_t = -cA_x \quad (A.33) \quad \text{into} \quad (A.51)
\]

\[
2 \epsilon AA_x \hat{\rho} + \mu^2 A_{xxx} \hat{\rho} = \frac{1}{c} [A_t \hat{\rho} - A_x \bar{\rho}_z \hat{\psi} - 2 \epsilon AA_x \bar{\rho}_z \hat{\psi} - \mu^2 A_{xxx} \bar{\rho}_z \hat{\psi} + \epsilon AA_x \bar{\rho}_z \hat{\psi} - \epsilon AA_x (\hat{\rho} \hat{\psi}_z)_z]
- \epsilon AA_x (\hat{\rho} (\hat{\psi}_z)_z)^2 - \epsilon AA_x (\hat{\rho} \hat{\psi}_{zz})_z + \mu^2 A_{zzt} \hat{\rho} \hat{\psi} = 0 \quad (A.52)
\]

Multiplying out the first term of (A.50) and reducing our two equations into a single equation by substituting (A.52) for the bold-face terms in (A.50) results in

\[
A_t \bar{\rho}_z \hat{\psi}_z + 2 \epsilon AA_t \bar{\rho}_z \hat{\psi}_z + \mu^2 A_{zzt} \bar{\rho}_z \hat{\psi}_z + A_t \bar{\rho}_z \hat{\psi}_{zz} + 2 \epsilon AA_t \bar{\rho}_z \hat{\psi}_{zz}
+ \mu^2 A_{zzt} \bar{\rho}_z \hat{\psi}_{zz} - A_x \hat{\rho} - \frac{1}{c} [A_t \hat{\rho} - A_x \bar{\rho}_z \hat{\psi} - 2 \epsilon AA_x \bar{\rho}_z \hat{\psi}]
- \mu^2 A_{xxx} \bar{\rho}_z \hat{\psi} + \epsilon AA_x \bar{\rho}_z \hat{\psi} - \epsilon AA_x (\hat{\rho} \hat{\psi}_z)_z + \epsilon AA_x (\hat{\rho} \hat{\psi}_{zz})_z + \mu^2 A_{zzt} \hat{\rho} \hat{\psi} = 0 \quad (A.53)
\]

As with the density equation, we substitute the nondimensional forms of equations (A.31), (A.33) and (A.34)

\[
\hat{\rho} = -\frac{\bar{\rho}_z}{c} \hat{\psi} \quad A_t = -cA_x \quad (\hat{\rho} \hat{\psi}_z) = \frac{\bar{\rho}_z}{c^2} \hat{\psi}
\]

and use the chain rule to gather terms involving \( \hat{\psi} \) and \( \hat{\rho} \) which are of the same order. This gives
A.5. The Korteweg-de Vries equation

\[
(A_t + cA_x) \left\{ \frac{2\hat{\rho}z\hat{\psi}}{c^3} \right\} + cAA_x \left\{ -2 \left( \hat{\rho}\hat{\psi}_z \right)_z + \frac{2\hat{\rho}z\hat{\psi}}{c^2} + \frac{1}{c} \left( \hat{\rho}\hat{\psi}\hat{\psi}_z \right)_z \right. \\
+ \left. \frac{1}{c} \left[ \hat{\rho} \left( \hat{\psi}_z \right)^2 - \hat{\rho}\hat{\psi}\hat{\psi}_{zz} \right]_z - \frac{1}{c^3} \hat{\rho}\hat{\psi}\hat{\psi}^2 \right\} + \mu^2 A_{xxx} \left\{ - \left( \hat{\rho}\hat{\psi}_z \right)_z + \frac{\rho_z\hat{\psi}}{c^2} - \rho\hat{\psi} \right\} = 0 \quad (A.55)
\]

If two new variables \( \alpha \) and \( \beta \) are defined to accommodate the terms in the curly brackets, (A.55) can be simplified to the well known KdV equation

\[
A_t = -cA_x + c\alpha AA_x + \mu^2 \beta A_{xxx} \quad (A.56)
\]

Benney (1966) has shown that through the application of appropriate boundary conditions and successive integration by parts that \( \alpha \) and \( \beta \) can be simplified to

\[
\alpha = -\frac{3}{2} \frac{\int_0^H \rho \hat{\psi}_z^2 dz}{\int_0^H \hat{\rho} \hat{\psi}_z^2 dz} \quad \beta = -\frac{\epsilon}{2} \frac{\int_0^H \rho \hat{\psi}_z^2 dz}{\int_0^H \hat{\rho} \hat{\psi}_z^2 dz} \quad (A.57)
\]

In the linear nondispersive limit (A.56) reduces to our linear long wave solution (A.33). The third term in (A.56) represents finite amplitude effects (nonlinear steepening) and the fourth term describes dispersive spreading which results from the wavelength being much greater than the water depth. For \( \epsilon \gg \mu^2 \) nonlinear steepening of the front face generates a discontinuous wave front or internal surge. If \( \epsilon \approx \mu^2 \ll 1 \) a balance may be achieved between nonlinear steepening and dispersive spreading which results in waves of unchanging form. In this case two solutions are possible – periodic cnoidal waves and solitary waves or solitons. Waves which may be approximated by the internal solitary wave solution are often observed in geophysical scale flows (e.g. Apel et al., 1985; Horn et al., 2000).

A.5.1 The solitary wave solution\

Since solitary waves are unchanging in form as they propagate we may evaluate a solitary wave solution by transforming (A.56) to a Lagrangian coordinate system moving at the solitary wave speed and then search for solutions which are stationary in this frame. We assume a Lagrangian (traveling) wave solution of the form \( A = f(x - ct) \), where \( c \approx c_0 + \epsilon c_1 \) is the sum of the linear long wave speed and the adjustment resulting from first order nonlinear and dispersive effects, respectively, and that nonlinear steepening is balanced by dispersion (i.e., \( \epsilon = \mu^2 \)). Under these assumptions equation (A.56) reduces to
Appendix A. Two-dimensional internal wave equations

\[ A_t = \alpha AA_x + \beta A_{xxx} \]  \hspace{1cm} (A.58)

Substitution of our traveling wave solution results in

\[ -cf' = \alpha ff' + \beta f''' \]  \hspace{1cm} (A.59)

which may be integrated once to yield

\[ -cf = \frac{\alpha}{2} f^2 + \beta f'' - C_1 \]  \hspace{1cm} (A.60)

Using \( f' \) as an integrating factor we integrate again to get

\[ (f')^2 = -\frac{\alpha}{3\beta} f^3 - \frac{c}{\beta} f^2 + C_1 f + C_2 \]  \hspace{1cm} (A.61)

Applying the boundary conditions \( f, f' \) and \( f'' \rightarrow 0 \) as \((x - ct) \rightarrow \pm \infty\) we see that \( C_1 \) and \( C_2 \) are both zero. Equation (A.61) may be integrated again using the substitution \( f = a \text{sech}^2 \theta \) (see Benney, 1966) to give the solitary wave solution for each vertical mode \( n \) as

\[ A_n = f(x - c_n t) = a_n \text{sech}^2 \left[ \frac{(x - c_n t)}{\lambda_n} \right] \]  \hspace{1cm} (A.62)

where \( a_n \) multiplied by \( \hat{\psi}_n \) or \( \hat{\rho}_n \) gives the perturbation amplitudes of the vertical velocity and density field, respectively, and

\[ c_1 = \frac{a_n \alpha}{3}, \quad \lambda_n^2 = \frac{12 \beta}{a_n \alpha} \]  \hspace{1cm} (A.63)

A.5.2 The hydrostatic approximation A common approximation in hydrodynamic models is the hydrostatic approximation. As we shall see this approximation may have serious effects on the character of the waves which may be reproduced by these models. In our notation the hydrostatic approximation is \( p_z = -\rho g \) or \((\bar{\rho} + \hat{\rho})_z = -(\bar{\rho} + \hat{\rho})g \) which assumes that the local fluid velocity has no influence on the local pressure, hence the name hydrostatic. Application of the hydrostatic approximation to our governing equations results in the left hand side of equation (A.6) being equal to zero. Carrying this simplification through the subsequent
derivations we see that the terms in equation (A.44) which scale with $\mu^2$ are now also equal to zero. Neglecting these terms in the derivation of the KdV equation gives the expected result

$$A_t = -c A_x + \epsilon r A A_x$$ (A.64)

hence the effects of dispersion are eliminated. A balance between nonlinear steepening and dispersion is now unattainable which results in artificial steepening of the wave profile (e.g. Daily, 1997) and precludes the nonlinear evolution of solitary waves from a larger scale disturbance (e.g. Horn et al., 1999).
Appendix A. Two-dimensional internal wave equations
Linear wave equation initial value problem for a two-layer stratified rectangular tank

B.1 Abstract

In Chapter 3, a particular solution of the linear wave equation is presented for the case of a rectangular tank stratified with two superposed fluids of differing density. The details of this solution are presented below.

B.2 Solution of the initial value problem

Consider baroclinic motion in a two-layer stratified rectangular tank subject to the Boussinesq and rigid lid approximations. If the amplitude of the motion of the layer interface is not too large, then \( \eta(x, t) \) satisfies

\[
\frac{c_o^2}{\eta_{xx}} = \eta_{tt} \tag{B.1}
\]

where \( c_o = (g' h_1 h_2 / (h_1 + h_2))^{1/2} \) is the linear wave speed (Gill, 1982, pg. 127).

Applying boundary conditions

\[
\eta_x(0, t) = 0 \tag{B.2}
\]

\[
\eta_x(l, t) = 0 \tag{B.3}
\]

\[
t \geq 0 \tag{B.4}
\]

and initial conditions

\[
\eta(x, 0) = f(x) = \frac{2\eta_o}{L} x - \eta_o \tag{B.5}
\]

\[
\eta_t(x, 0) = 0 \tag{B.6}
\]

\[
0 \leq x \leq L. \tag{B.7}
\]

Applying separation of variables we assume that

\[
\eta(x, t) = X(x)T(t) \tag{B.8}
\]

\[
\eta_t(x, t) = X(x)T'(t)
\]

\[
\eta_{tt}(x, t) = X(x)T''(t)
\]

\[
\eta_x(x, t) = X'(x)T(t)
\]

\[
\eta_{xx}(x, t) = X''(x)T(t)
\]
Appendix B. Linear wave equation initial value problem

and substitute into (B.1) to obtain

\[ \frac{X''}{X} = \frac{1}{c_o^2} \frac{X''}{X} = -\sigma \]  \hspace{1cm} (B.9)

where \( \sigma \) is a constant. We find that \( X \) and \( T \) may now be separated as

\[ X'' + \sigma X = 0 \]  \hspace{1cm} (B.10)
\[ T'' + c_o^2 \sigma T = 0. \]  \hspace{1cm} (B.11)

We now apply the boundary conditions to determine the permissible values of \( \sigma \). Recasting the boundary conditions in terms of \( X \) and \( T \)

\[ \eta_x(x, t) = X'(x)T(t) \]  \hspace{1cm} (B.12)
\[ \eta_x(0, t) = X'(0)T(t) = 0. \]  \hspace{1cm} (B.13)

Therefore \( X'(0) = 0 \) for a nontrivial solution over all \( t \). Similarly,

\[ \eta_x(l, t) = X'(l)T(t) = 0 \]  \hspace{1cm} (B.14)

and \( X'(l) = 0 \).

If \( \sigma < 0 \), let \( \sigma = -\lambda^2 \) and (B.10) becomes

\[ X'' - \lambda^2 X = 0 \]  \hspace{1cm} (B.15)

which has a general solution

\[ X(x) = k_1 \sinh \lambda x + k_2 \cosh \lambda x \]  \hspace{1cm} (B.16)

that can only satisfy the boundary conditions if \( k_1 = k_2 = 0 \) (Boyce & DiPrima, 1992, pg. 547).

If \( \sigma = 0 \),

\[ X'' = 0 \]  \hspace{1cm} (B.17)
\[ X' = k_1 \]  \hspace{1cm} (B.18)
\[ X = k_1 x + k_2 \]  \hspace{1cm} (B.19)
so

\[ X'(0) = k_1 = 0 \quad (B.20) \]
\[ X'(l) = k_1 = 0 \quad (B.21) \]
\[ X'(l) = k_1 = 0 \quad (B.22) \]

and

\[ X(x) = k_2. \quad (B.23) \]

If \( \sigma > 0 \), let \( \sigma = \lambda^2 \) and (B.10) becomes

\[ X'' + \lambda^2 X = 0 \quad (B.24) \]

which has a general solution

\[ X(x) = k_1 \cos \lambda x + k_2 \sin \lambda x \quad (B.25) \]
\[ X'(x) = \lambda (k_2 \cos \lambda x - k_1 \sin \lambda x). \quad (B.26) \]

Applying the boundary conditions

\[ X'(0) = \lambda k_2 = 0 \rightarrow k_2 = 0 \quad (B.27) \]
\[ X'(l) = -k_1 \lambda \sin \lambda l = 0 \rightarrow \sin \lambda l = 0 \quad (B.28) \]

so \( \lambda l = n\pi, n = 1, 2, 3, \ldots \) and (B.25) becomes

\[ X(x) = k_1 \cos \frac{n\pi}{l} x. \quad (B.29) \]

We now have to check that the solutions (B.23) and (B.29) also satisfy (B.11).

With \( \sigma = 0 \)

\[ T'' = 0 \quad (B.30) \]
\[ T' = k_1 \quad (B.31) \]
\[ T = k_1 t + k_2 \quad (B.32) \]
and from (B.5), (B.6) and (B.8)

\[ \eta(x, 0) = X(x)T'(0) = 0 \]  \hspace{1cm} (B.33)

\[ \eta(x, 0) = X(x)T(0) = f(x) \]  \hspace{1cm} (B.34)

With \( \sigma > 0 \), \( \sigma = \lambda^2 = (n\pi/l)^2 \) and (B.11) becomes

\[ T'' + \left( \frac{c_\sigma n\pi}{l} \right)^2 T = 0. \]  \hspace{1cm} (B.35)

That has a general solution

\[ T(t) = k_1 \sin \left( \frac{c_\sigma n\pi}{l} t \right) + k_2 \cos \left( \frac{c_\sigma n\pi}{l} t \right). \]  \hspace{1cm} (B.36)

Combining solutions (B.29) and (B.36) according to (B.8)

\[ \eta(x, t) = \sum_{n=1}^{\infty} \cos \left( \frac{n\pi}{l} x \right) \left[ k_1 \sin \left( \frac{c_\sigma n\pi}{l} t \right) + k_2 \cos \left( \frac{c_\sigma n\pi}{l} t \right) \right]. \]  \hspace{1cm} (B.37)

We can find the constants by applying the initial conditions

\[ \eta_t(x, t) = \sum_{n=1}^{\infty} \cos \left( \frac{n\pi}{l} x \right) \left[ c_\sigma n\pi k_1 \cos \left( \frac{c_\sigma n\pi}{l} t \right) - k_2 \sin \left( \frac{c_\sigma n\pi}{l} t \right) \right] \]  \hspace{1cm} (B.38)

\[ \eta_t(x, 0) = \sum_{n=1}^{\infty} \frac{c_\sigma n\pi k_1}{l} \cos \left( \frac{n\pi}{l} x \right) = 0 \]  \hspace{1cm} (B.39)

which requires that \( k_1 = 0 \) for all \( n \) (Boyce & DiPrima, 1992, pg. 556). The other constant is found by applying the other initial condition

\[ \eta(x, t) = \sum_{n=1}^{\infty} k_2 \cos \left( \frac{n\pi}{l} x \right) \cos \left( \frac{c_\sigma n\pi}{l} t \right) \]  \hspace{1cm} (B.40)

\[ \eta(x, 0) = \sum_{n=1}^{\infty} k_2 \cos \left( \frac{n\pi}{l} x \right) = f(x). \]  \hspace{1cm} (B.41)

Consequently, \( k_2 \) must be the coefficients in the Fourier cos series of period \( 2l \) for \( f \), and are given by Boyce & DiPrima (1992)
\[ k_2 = \frac{2}{l} \int_0^l f(x) \cos \left( \frac{n\pi}{l} x \right) dx \quad n = 1, 2, 3, \ldots \] \hspace{1cm} (B.42)

\[ k_2 = \frac{2}{l} \int_0^l \left( \frac{2\eta_o}{L} x - \eta_o \right) \cos \left( \frac{n\pi}{l} x \right) dx \quad n = 1, 2, 3, \ldots \] \hspace{1cm} (B.43)

The general solution subject to the boundary and initial conditions is therefore

\[ \eta(x, t) = \sum_{n=1}^{\infty} k_2 \cos \left( \frac{n\pi}{l} x \right) \cos \left( \frac{c_n n\pi}{l} t \right), \] \hspace{1cm} (B.44)

where

\[ k_2 = \frac{2}{l} \int_0^l \left( \frac{2\eta_o}{L} x - \eta_o \right) \cos \left( \frac{n\pi}{l} x \right) dx \quad n = 1, 2, 3, \ldots \] \hspace{1cm} (B.45)

\[ = -8 \frac{\eta_o}{(n\pi)^2}. \] \hspace{1cm} (B.46)

The solution is plotted in figure (3.2).
Appendix B. Linear wave equation initial value problem
Bibliography


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